

Complex Hyperbolic Triangle Groups

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Chapter 0

Introduction

In this thesis we study the discreteness criteria for complex hyperbolic triangle groups, generated by reflections in the complex hyperbolic 2-space $\mathbf{H}_{\mathbb{C}}^2$.

The question of classification of discrete subgroups in the projective unitary group $PU(n, 1)$ remains widely open. $PU(n, 1)$ is the holomorphic isometry group of the n -dimensional complex hyperbolic space $\mathbf{H}_{\mathbb{C}}^n$. Complex hyperbolic space is an example of a non-compact symmetric space of rank 1. There are three other classes of rank 1 symmetric spaces of non-compact type. They are the real hyperbolic space $\mathbf{H}_{\mathbb{R}}^n$, the quaternionic hyperbolic space $\mathbf{H}_{\mathbb{H}}^n$ and the octonionic hyperbolic plane $\mathbf{H}_{\mathbb{O}}^2$.

Discrete subgroups of finite covolume (lattices) in $\mathbf{H}_{\mathbb{C}}^2$ are isolated points in a large space. Logically, one would expect first a classification of lattices which are sparse compared with their infinite covolume complement.

A fundamental problem in the study of symmetric spaces is the relationship between arithmetic groups and lattices. Borel and Harish-Chandra [1] have shown that in all symmetric spaces of non-compact type, all arithmetic groups are lattices. Gromov and Piatetski-Shapiro [12] have given examples of non-arithmetic lattices in $\mathbf{H}_{\mathbb{R}}^n$ for all $n \geq 2$. Corlette [3] and Gromov and Schoen [13] have shown that in $\mathbf{H}_{\mathbb{H}}^n$ for $n \geq 2$ and in $\mathbf{H}_{\mathbb{O}}^2$ all lattices are arithmetic. Furthermore, Margulis

[18] proved that lattices in the isometry group of any symmetric space of rank at least 2 are arithmetic. Complex hyperbolic space is the only class of symmetric spaces of non-compact type where this question has not been settled.

In 1980, Mostow [20] constructed examples of lattices in $\mathbf{H}_{\mathbb{C}}^2$ from triangle groups and gave the first examples of non-arithmetic lattices in $PU(2, 1)$.

A triangle group is a group generated by reflections in the sides of a geodesic triangle, classically these could be Euclidean, spherical or hyperbolic triangles. Consider the real hyperbolic plane $\mathbf{H}_{\mathbb{R}}^2 \cong \mathbf{H}_{\mathbb{C}}^1$ (or Poincaré disc). One of the simplest types of discrete groups acting on $\mathbf{H}_{\mathbb{R}}^2$ are the Fuchsian triangle groups. These groups are generated by the reflections in the sides of a geodesic triangle in this plane. If we choose the triangle to have angles $\pi/p_1, \pi/p_2, \pi/p_3$ with $p_i \in \mathbb{N} \cup \{\infty\}$ and provided $1/p_1 + 1/p_2 + 1/p_3 < 1$ then the group will be a discrete subgroup of $PSL(2, \mathbb{R})$ and the triangle is a fundamental domain for its action on the plane.

In the real hyperbolic case the triangle, and hence the group, is completely determined up to isometry by its angles. We thus have a countable family of non-isometric triangles whose reflection groups are non-conjugate in $PSL(2, \mathbb{R})$. However, Takeuchi proved there is only a finite number of arithmetic lattices among these groups [32] and gave a complete list, the most famous example being the modular group $PSL(2, \mathbb{Z})$ which is contained with index 2 in the triangle group with $(p_1, p_2, p_3) = (2, 3, \infty)$. Since there are infinitely many triangle groups acting discretely on $\mathbf{H}_{\mathbb{R}}^2$ with finite covolume, triangle groups are generically non-arithmetic.

We define a triangle in $\mathbf{H}_{\mathbb{C}}^2$ as three distinct complex slices. If two slices intersect in $\mathbf{H}_{\mathbb{C}}^2$ then the angle at this vertex is defined as the infimum of the angles between the real geodesics contained in the complex geodesics and passing through the intersection point. We can carry out a similar construction to that of $\mathbf{H}_{\mathbb{R}}^2$ in $\mathbf{H}_{\mathbb{C}}^2$, that is choose a triangle to have angles $\pi/p_1, \pi/p_2, \pi/p_3$ with

$p_i \in \mathbb{N} \cup \{\infty\}$ and with the provision $1/p_1 + 1/p_2 + 1/p_3 < 1$. Note that this works for $p_1, p_2, p_3 \in \mathbb{Q}^+$ as well. Then a complex hyperbolic triangle group (the prefix ‘complex hyperbolic’ may sometimes be dropped in the notation when it is clear what type of triangle group we are talking about) is a group generated by three complex reflections, denoted ι_1 , ι_2 and ι_3 , in the three complex geodesics connecting the pairs of vertices of the triangle. A particular difficulty with complex hyperbolic triangle groups is that a complex hyperbolic triangle is not uniquely determined by its angles. More precisely, there is a one-dimensional deformation space of non-isometric complex hyperbolic triangles with a fixed triple of angles, distinguished by one real parameter. We call this parameter the angular invariant, described in section 1.2.

Also unlike real reflections, complex reflections can be of arbitrary order. Mostow had studied new examples of non-arithmetic lattices in $\mathbf{H}_{\mathbb{C}}^2$ generated by three complex reflections of order 3, 4 and 5. Although much of the literature since Mostow is confined to the case where the reflections have order 2, there has been work on higher order reflections by Pratoševitch [27] and Parker and Paupert [25]. In this thesis we only consider the case of order 2 complex reflections.

Mostow’s first examples in $PU(2, 1)$ were constructed by giving explicit generators, and verifying that the corresponding groups are discrete by building fundamental polyhedra for their action. There are very few values of the angular invariant for which these triangle groups are discrete. For these values Mostow shows discreteness using Dirichlet domains, determined by computer experimentation.

The Dirichlet polyhedron $D_{\Gamma}(z_0)$ for a discrete subgroup $\Gamma \leq PU(n, 1)$, based at a ‘central’ point $z_0 \in \mathbf{H}_{\mathbb{C}}^n$, is the set of points in $\mathbf{H}_{\mathbb{C}}^n$ that are closer to z_0 than to any other point in the Γ -orbit of z_0 . Assuming that z_0 is not fixed by any non-trivial element of Γ then $D_{\Gamma}(z_0)$ is a fundamental domain for Γ . This

approach has the advantage of generality and simplicity in principle, but it is very hard to use in practice. Namely, the basic object in this construction is the boundary of $D_\Gamma(z_0)$ which consists of pieces of hypersurfaces in $\mathbf{H}_\mathbb{C}^n$ contained in bisectors, that is the locus of points equidistant from z_0 and one of its images. In the case of real hyperbolic geometry, groups generated by reflections are well understood. The main reason for this is due to the fact there are totally geodesic real hypersurfaces in $\mathbf{H}_\mathbb{R}^n$. In complex hyperbolic space $\mathbf{H}_\mathbb{C}^n$, or in any space where sectional curvature is not constant, there are no totally geodesic real hypersurfaces, hence it is difficult to control the intersections of bisectors. This situation makes it difficult to construct not only discrete subgroups of $PU(n, 1)$, but also fundamental polyhedra for such groups. One of the goals of Goldman's book [11] was to understand the intersections of bisectors.

There were some minor errors in Mostow's construction; see Deraux [5]. An alternative construction of fundamental domains for the same groups was given by Deraux, Falbel and Paupert [7]. Later Livné, Mostow, Deligne, Thurston and Deraux constructed more examples of lattices that are generated by three complex reflections. More recently Deraux, Paupert and Parker [8] and Thompson [33] have found new candidates for non-arithmetic lattices from triangle groups.

Different constructions of hypersurfaces have appeared, the nature of which are linked to the type of the group in question. In his 2002 survey Schwartz [30] described some different examples. One may also construct lattices in complex hyperbolic space in different ways. Nevertheless, it is often difficult to do this and there are relatively few known constructions. For more details on methods of constructing lattices and an overview of this area see Parker's survey [24].

Deligne and Mostow [4] found a non-arithmetic lattice in $\mathbf{H}_\mathbb{C}^3$, but for $\mathbf{H}_\mathbb{C}^n$ with $n \geq 4$ the question is open. The fact that there are relatively few known constructions of complex hyperbolic lattices may well account for the fact that this problem is still open. But the question of classifying all such discrete groups

is not restricted to high dimensions of complex hyperbolic space, a lot is still unknown for complex hyperbolic space of dimension 2. This is our motivation for studying discreteness of complex hyperbolic triangle groups in $\mathbf{H}_{\mathbb{C}}^2$.

In 1992, Goldman and Parker [10] were the first to study triangle groups for the whole deformation space of non-isometric complex hyperbolic triangles with fixed angles. They considered the ideal triangle groups (triangles with vertices on the boundary) and looked at discrete and faithful representations of ideal triangle groups into $PU(2, 1)$. Here they conjectured that the ideal triangle group representations were discrete and faithful if and only if the product of the three generators, denoted by $w_B = \iota_1 \iota_2 \iota_3$, was not elliptic. They had already shown that there was a smaller interval in the parameter space for which ideal triangle groups are discrete and faithful. Schwartz [29] then proved a stronger version of this conjecture, namely that the ideal triangle group representations are discrete and faithful if and only if w_B is not elliptic and are not discrete otherwise.

Since then there has been much work following Goldman and Parker by investigating discrete and faithful representations; we have already mentioned Schwartz [29] for example. Also there has been the study of discrete representations where certain group elements are elliptic of finite order, hence the representations are necessarily unfaithful; see Schwartz [31], Deraux [6], Falbel and Parker [9], Parker [22] for example.

In the first part of this thesis we follow a similar route to the latter, namely in the group with angles $\pi/p_1, \pi/p_2, \pi/p_3$ we consider the case when two elements $w_A := \iota_3 \iota_2 \iota_3 \iota_1$ and $w_B := \iota_1 \iota_2 \iota_3$ are elliptic. Then it is clear that for the triangle group to be discrete these two elements must be of finite order otherwise the group is necessarily non-discrete. We then reduce the question of finding out whether both elliptic elements could be of finite order simultaneously to solving equation (2.1.3) which is given as a rational linear combination of cosines of rational

multiples of π equal to one, shown below

$$\begin{aligned} 1 = & \cos\left(2\left(\frac{\pi}{p_1} + \frac{\pi}{p_2}\right)\right) + \cos\left(2\left(\frac{\pi}{p_1} - \frac{\pi}{p_2}\right)\right) - \cos\left(\frac{2\pi}{p_3}\right) \\ & - \cos x - \cos y - \cos z - \cos\left(\frac{2\pi}{n}\right), \end{aligned}$$

where $x, y, z \in \mathbb{Q}\pi$ and $n \in \mathbb{Z}$. For more details on the variables see section 2.

Now Conway and Jones [2] classified all rational linear combinations of at most four cosines of rational multiples of π which is rational and then Parker [21] extended the classification to all rational linear combinations of at most six cosines of rational multiples of π which vanish. This type of approach to find suitable candidates of triangle groups that could be discrete has been used by Parker [23]. He analysed all equilateral complex hyperbolic triangle groups with the product of three generators being elliptic of finite order using the result of Conway and Jones. This classic result of Conway and Jones has been used in many different areas of mathematics, for example Poone and Rubinstein [26] used it to find the number of interior intersections made by the diagonals of a regular n -gon.

For a general complex hyperbolic triangle group with angles π/p_1 , π/p_2 and π/p_3 as above, the question of whether the two elliptic elements are of finite order reduces to a rational linear combination of seven cosines. Therefore to use work that has already been done we look at certain types of triangles with a relation between the three angles so that the equation reduces in length. The first example we look at are triangle groups with angles π/p , π/p and $\pi/2p$ which reduces the equation to a rational linear combination of six cosines of rational multiples of π that vanishes. Thus we can use the list given by Parker to find solutions. A simplified version of the result that we obtain is

Theorem (2.2.0.19). *The complex hyperbolic triangle group of signature $(p, p, 2p; n)$ i.e. with angles π/p , π/p and $\pi/2p$, is non-discrete for $p \geq 13$ when both w_A and w_B are elliptic.*

Table 2.1 displays the results of this theorem in a much more digestible fashion.

We then consider the family of complex hyperbolic triangle groups with angles π/p_1 , π/p_2 and π/p_3 such that $1/p_1 = 1/p_2 + 1/p_3$, that is the family of triangles where the first angle is the sum of the remaining two. This reduces equation (2.1.3) to a rational linear combination of five cosines of rational multiples of π that equal 1.

As a result we can use the smaller list of Conway and Jones [2] to find solutions. Therefore using the list of Conway and Jones we then study in depth the triangle groups with angles π/p , $\pi/2p$, $\pi/2p$ and π/p , $2\pi/3p$, $\pi/3p$ respectively. We first consider integer values for p and prove the two following propositions:

Proposition (2.3.0.23). *The complex hyperbolic triangle group of signature $(p, 2p, 2p; n)$ i.e. with angles π/p , $\pi/2p$ and $\pi/2p$, is non-discrete for $p \geq 10$ when both w_A and w_B are elliptic.*

Proposition (2.3.0.28). *The complex hyperbolic triangle group of signature $(p, 3p/2, 3p; n)$ i.e. with angles π/p , $2\pi/3p$ and $\pi/3p$, is non-discrete for $p \geq 11$ when both w_A and w_B are elliptic.*

We display these propositions succinctly in Table 2.2 and Table 2.3. Next we prove a similar result for a general triangle group in this family of complex hyperbolic triangle groups. To do this we again look at the triangle groups with angles π/p , $\pi/2p$, $\pi/2p$ and π/p , $2\pi/3p$, $\pi/3p$ but consider when p is rational and notice a distinct similarity between the two proofs and use this similarity to generalise the result. A simplified version of the main Theorem in this section is

Theorem (2.3.0.38). *The complex hyperbolic triangle group with angles $\pi/p_1 > \pi/p_2 \geq \pi/p_3$ such that $1/p_1 = 1/p_2 + 1/p_3$ is non-discrete for $12 \leq p_1 \in \mathbb{Z}$, $p_2, p_3 \in \mathbb{Q}$ when both w_A and w_B are elliptic and w_A has integer order n .*

To solve an equation of a rational linear combination of seven cosines of rational multiples of π equal to one in general, we would need to extend the workings of

Conway and Jones [2] and Parker [21]. We do exactly this and generalise the work of Conway and Jones and Parker to seven distinct rational linear combinations of cosines of rational multiples of π that is rational, see Theorem 2.4.3.1. With our list it is possible to check, more than likely with the aid of a computer, for solutions to equation (2.1.3) for a general complex hyperbolic triangle with angles $\pi/p_1, \pi/p_2, \pi/p_3$ where both elements w_A and w_B are elliptic. Hence in principle, one can sieve through all non-discrete triangle groups of this type and leave only those, if any, that are a suitable candidate to be discrete. We leave this problem open.

Finally we look at ultra parallel complex hyperbolic triangle groups. An ultra parallel complex hyperbolic triangle is a complex hyperbolic triangle where the sides of the triangle do not intersect inside $\mathbf{H}_{\mathbb{C}}^2$. Therefore, instead of the angle between the two geodesics we look at the distance between them, where the distance function is defined in section 1.1.2. Similar to triangles with vertices inside $\mathbf{H}_{\mathbb{C}}^2$, an ultra parallel complex hyperbolic triangle is determined up to isometry by the three distances between the complex geodesics, say l_1, l_2, l_3 , and one real parameter α

We introduce criteria for discreteness using a method called compressing. The technique uses a ‘ping pong’ style approach to show discreteness of a group. This discreteness criterion is a close variant of the Ping-Pong Lemma and the Klein combination theorem. For original sources, see [17] and [19]. A similar method was used by Schwartz [29] to prove the Goldman-Parker conjecture. Wyss-Galifent [34] used this approach in studying the ultra parallel triangle groups with distances $l, l, 2l$ and $l, l, 0$ apart and obtained partial results similar to that of Goldman and Parker [10] for the ideal case. We study the ultra parallel complex hyperbolic triangle group with distances $l_1, l_2, 0$ and use this ‘ping-pong’ method on the boundary $\partial\mathbf{H}_{\mathbb{C}}^2$ of the complex hyperbolic space and give an area of discreteness for such triangle groups:

Theorem (3.3.0.7). *Let $0 \leq l_1, l_2 \in \mathbb{R}$. An ultra-parallel triangle group with distances $l_1, l_2, 0$ is discrete for*

$$\sin \frac{\alpha}{2} \geq \left(\cosh \left(\frac{l_1}{2} \right) + \cosh \left(\frac{l_2}{2} \right) \right)^{-1}.$$

Chapter 1

Preliminaries

In this chapter we introduce some basic notions, full details about the material in this chapter can be found in [11]. We give a description of the complex hyperbolic space $\mathbf{H}_{\mathbb{C}}^2$ using the projective model and identify the boundary of the complex hyperbolic space $\partial\mathbf{H}_{\mathbb{C}}^2$ with the one point compactification of the Heisenberg group \mathcal{H} . We then introduce the metric on $\mathbf{H}_{\mathbb{C}}^2$ and subsequently the isometry group. We discuss different types of isometries we have and how we can determine the type by using the trace formula introduced in Goldman's book [11]. We define complex geodesics and describe how to determine the type of intersection between two complex geodesics. The intersection of a complex geodesic with the boundary is called a chain, we give a representation of some chains in \mathcal{H} .

Given complex geodesic C , there is a unique isometry of order 2 called a reflection, whose fixed point set is equal to C . In section 1.2 we introduce a complex hyperbolic triangle as a triple of complex geodesics and define a complex hyperbolic triangle group Γ as a group generated by order 2 reflections in three complex geodesics. We discuss the two main types of complex hyperbolic triangles, those with all vertices inside $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$ and those with vertices outside (ultra parallel). We then describe a parametrisation of the real one dimensional space of non-isometric complex hyperbolic triangles with fixed angles (resp. distances)

in $\mathbf{H}_{\mathbb{C}}^2$ using the angular invariant α .

1.1 Complex Hyperbolic 2-Space

One description of the *complex hyperbolic 2-space* $\mathbf{H}_{\mathbb{C}}^2$ is given by the projective model. Here we take an Hermitian form of signature $(2, 1)$ on \mathbb{C}^3 and projectivise the set of complex lines on which this Hermitian form is negative. Let $\mathbb{C}^{2,1}$ be the complex vector space of (complex) dimension 3 equipped with a non-degenerate Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(2, 1)$. We can describe $\langle \cdot, \cdot \rangle$ by a non-singular 3×3 Hermitian matrix J , with 2 positive eigenvalues and 1 negative eigenvalue.

Given two Hermitian forms J and J' with the same signature, we can pass between them using a Cayley transform C . That is, we can write, in a non-unique way,

$$J' = C^* J C.$$

This means that one does not necessarily need to specify the form, but only the signature. The choice of a particular Hermitian form is equivalent to a choice of basis in $\mathbb{C}^{2,1}$. The Hermitian form which we will be concerned with is defined to be

$$\langle z, w \rangle = w^* J z = z_1 \bar{w}_1 + z_2 \bar{w}_2 - z_3 \bar{w}_3,$$

where $z = (z_1, z_2, z_3)^t$ and $w = (w_1, w_2, w_3)^t$. It is given by the Hermitian matrix

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

1.1.1 The Projective Model of the Complex Hyperbolic Space

If $z \in \mathbb{C}^3$ then $\langle z, z \rangle \in \mathbb{R}$ as $\langle z, z \rangle = \overline{\langle z, z \rangle}$. Therefore we can define subsets of $\mathbb{C}^{2,1}$ by

$$\begin{aligned} V_- &= \{z \in \mathbb{C}^{2,1} | \langle z, z \rangle < 0\}, \\ V_0 &= \{z \in \mathbb{C}^{2,1} \setminus \{0\} | \langle z, z \rangle = 0\}, \\ V_+ &= \{z \in \mathbb{C}^{2,1} | \langle z, z \rangle > 0\}. \end{aligned}$$

We say that $z \in \mathbb{C}^{2,1}$ is *negative*, *null* or *positive* if z is in V_- , V_0 or V_+ respectively. Now for any non-zero complex scalar λ , the point λz is negative, null or positive if and only if z is. This is because $\langle \lambda z, \lambda z \rangle = |\lambda|^2 \langle z, z \rangle$. So we define a projection map \mathbb{P} , on those points of $\mathbb{C}^{2,1}$ with $z_3 \neq 0$ by:

$$\mathbb{P} : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{pmatrix} z_1/z_3 \\ z_2/z_3 \end{pmatrix} \in \mathbb{C}^2.$$

Remark 1.1.1.1. Note that \mathbb{P} is well defined on $V_- \cup V_0$ since the Hermitian form $\langle \cdot, \cdot \rangle$ is positive definite on $\{z \in \mathbb{C}^3 | z_3 = 0\}$.

Definition. The projective model of the complex hyperbolic space is defined to be the collection of negative lines in $\mathbb{C}^{2,1}$ and its ideal boundary is defined to be the collection of null lines. Namely, $\mathbf{H}_{\mathbb{C}}^2$ is $\mathbb{P}(V_-)$ and $\partial \mathbf{H}_{\mathbb{C}}^2$ is $\mathbb{P}(V_0)$.

We consider the section of $\mathbb{C}^{2,1}$ defined by $z_3 = 1$, that is we take the column vector $z = (z_1, z_2, 1)^t \in \mathbb{C}^{2,1}$ and find what it means for $\langle z, z \rangle$ to be negative, null or positive. A point $z = (z_1, z_2) \in \mathbf{H}_{\mathbb{C}}^2$ satisfies $|z_1|^2 + |z_2|^2 < 1$ and is in the open unit 4-ball $\mathbb{B}^2 \subset \mathbb{C}^2$. A point $z = (z_1, z_2) \in \partial \mathbf{H}_{\mathbb{C}}^2$ is in the boundary of the unit ball, which is the sphere S^3 given by $|z_1|^2 + |z_2|^2 = 1$. We call this model of $\mathbf{H}_{\mathbb{C}}^2$

the *ball model* and it is analogous to the Klein model of the real hyperbolic plane.

Definition. We say that the standard lift of a point $z = (z_1, z_2) \in \mathbf{H}_{\mathbb{C}}^2$ to $\mathbb{C}^{2,1}$ is the column vector $z = (z_1, z_2, 1)^t$.

In the same way that the boundary of the real hyperbolic space is the one point compactification of the Euclidean space of one dimension lower, we may identify the boundary $\partial\mathbf{H}_{\mathbb{C}}^2$ with $\mathcal{H} = \mathbb{C} \times \mathbb{R} \cup \{\infty\}$, a one point compactification of the Heisenberg group.

Definition. The Heisenberg group is the set of all pairs $(\zeta, v) \in \mathbb{C} \times \mathbb{R}$ with the non-abelian group law

$$(\zeta_1, v_1) * (\zeta_2, v_2) = (\zeta_1 + \zeta_2, v_1 + v_2 + \text{Im}(\zeta_1 \bar{\zeta}_2)).$$

The Heisenberg group is isomorphic to the group of matrices of the form

$$\begin{bmatrix} 1 & \bar{\zeta} & \frac{1}{2}|\zeta|^2 + iv \\ 0 & 1 & \zeta \\ 0 & 0 & 1 \end{bmatrix}.$$

One such homeomorphism taking S^3 to \mathcal{H} is given by the stereographic projection:

$$\begin{aligned} \mathcal{P} &: S^3 \longrightarrow \mathcal{H} \\ (z_1, z_2) &\mapsto \left(\frac{z_1}{1 + z_2}, -\text{Im} \left(\frac{1 - z_2}{1 + z_2} \right) \right) \\ (0, -1) &\mapsto \infty \end{aligned}$$

We denote the image of $z \in \mathbb{C}^{2,1}$ under the projectivisation map \mathbb{P} by $[z]$. For the projective model, the metric on $\mathbf{H}_{\mathbb{C}}^2$ is given by the distance function $\rho(\cdot, \cdot)$ defined by the formula

$$\cosh^2 \left(\frac{\rho([z], [w])}{2} \right) = \frac{\langle z, w \rangle \langle w, z \rangle}{\langle z, z \rangle \langle w, w \rangle}$$

and called the *Bergman metric*.

1.1.2 Isometries

We look for a linear transformation \mathcal{A} such that $\langle \mathcal{A}(z), \mathcal{A}(w) \rangle = \langle z, w \rangle$ for all z and w . Now $\mathcal{A}(z) = Az$, where A is a 3×3 complex matrix, so

$$\langle \mathcal{A}(z), \mathcal{A}(w) \rangle = \langle Az, Aw \rangle = (Aw)^* JAz = w^* (A^* J A) z$$

and $\langle z, w \rangle = w^* J z$. Therefore,

$$\langle \mathcal{A}(z), \mathcal{A}(w) \rangle = \langle z, w \rangle \iff w^* (A^* J A) z = w^* J z.$$

If we take $z = e_l$ and $w = e_k$ as the standard basis vectors then each component $(A^* J A)_{k,l} = J_{k,l} \quad \forall k, l$. This implies that the matrices coincide, so for all z and w

$$\begin{aligned} \langle \mathcal{A}(z), \mathcal{A}(w) \rangle = \langle z, w \rangle &\iff A^* J A = J \\ &\iff A^{-1} = J^{-1} A^* J. \end{aligned}$$

If $A^{-1} = J^{-1} A^* J$ then A preserves the Hermitian form $\langle \cdot, \cdot \rangle$ as above and we say that $A \in U(2, 1)$ is a unitary matrix with respect to $\langle \cdot, \cdot \rangle$. Now any matrix in $U(2, 1)$ which is a non-zero complex scalar multiple of the identity matrix I , maps each line in $\mathbb{C}^{2,1}$ to itself and so acts trivially on the complex hyperbolic space. The projective unitary group $PU(2, 1)$ is the quotient of $U(2, 1)$ by the right multiplication of $U(1)$, where $U(1)$ is identified as the set of all scalar matrices $\{e^{i\theta} I \mid 0 \leq \theta < 2\pi\}$. Sometimes we wish to consider the unimodular unitary matrices and we denote the corresponding group by $SU(2, 1)$. The group $SU(2, 1)$ is a 3-fold cover of $PU(2, 1)$.

Since the Bergman metric is given in terms of the Hermitian form $\langle \cdot, \cdot \rangle$, a linear

transformation \mathcal{A} , which preserves the Hermitian form, acts isometrically on the projective model of the complex hyperbolic space. So $PU(2, 1)$ is a subgroup of the complex hyperbolic isometry group. The isometries coming from $PU(2, 1)$ are orientation preserving and holomorphic. There are no orientation reversing isometries of $\mathbf{H}_{\mathbb{C}}^2$.

However, there are isometries of $\mathbf{H}_{\mathbb{C}}^2$ not in $PU(2, 1)$, for example complex conjugation, which is anti-holomorphic. Now, every isometry of $\mathbf{H}_{\mathbb{C}}^2$ is either holomorphic or anti-holomorphic. Moreover, each holomorphic isometry of $\mathbf{H}_{\mathbb{C}}^2$ is given by a matrix in $PU(2, 1)$ and each anti-holomorphic isometry is given by complex conjugation followed by a matrix in $PU(2, 1)$.

Remark 1.1.2.1. *This means that we can use complex linear algebra to study the geometry of the complex hyperbolic space.*

Similar to real hyperbolic geometry we have three general types of holomorphic isometries. Given a holomorphic complex hyperbolic isometry A , it is said to be:

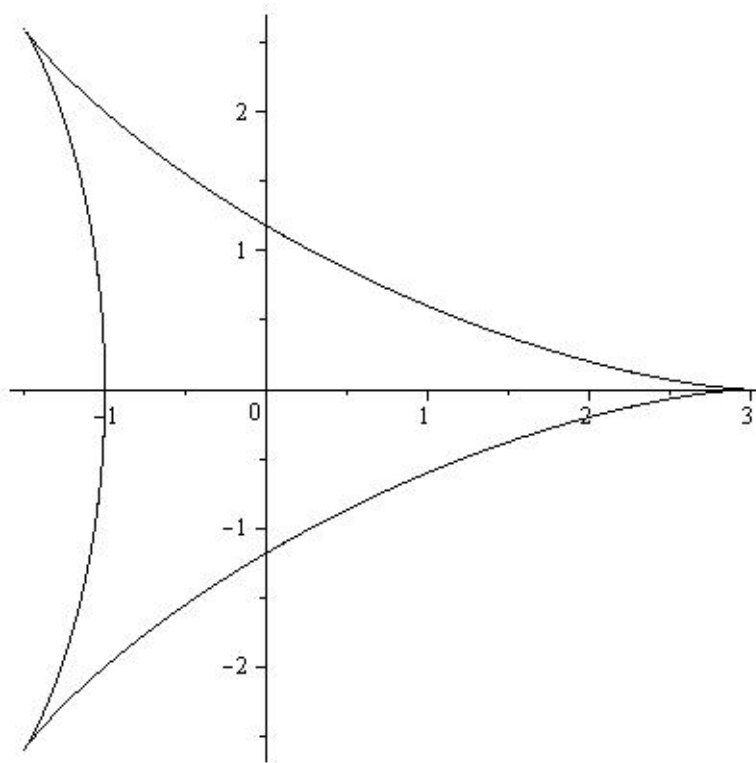
- i) *loxodromic* (or *hyperbolic*) if it has exactly two fixed points in $\partial\mathbf{H}_{\mathbb{C}}^2$
- ii) *parabolic* if it has exactly one fixed point in $\partial\mathbf{H}_{\mathbb{C}}^2$
- iii) *elliptic* if it has at least one fixed point in $\mathbf{H}_{\mathbb{C}}^2$

Now in the complex hyperbolic space we can refine this classification further. An elliptic element is called *regular elliptic* if all of its eigenvalues are distinct. An element whose eigenvalues are all 1 is called *unipotent* and with the exception of the identity, all unipotent elements are parabolic.

We can use the discriminant function introduced by William Goldman [11, Theorem 6.24],

$$f(z) = |z|^4 - 8\operatorname{Re}(z^3) + 18|z|^2 - 27, \quad (1.1.1)$$

to classify these isometries by the traces of the corresponding matrices.

Figure 1.1: The deltoid given by $f(z) = 0$

An isometry given by the matrix $A \in SU(2, 1)$ is regular elliptic if and only if $f(\text{trace } A) < 0$ and hyperbolic if and only if $f(\text{trace } A) > 0$. Now if $f(\text{trace } A) = 0$ then A is either a complex reflection in a complex geodesic or a complex reflection about a point or A is parabolic. Note that for $z \in \mathbb{R}$ the discriminant function factors into $f(z) = (z + 1)(z - 3)^3$. This means that a matrix $A \in SU(2, 1)$ with real trace is regular elliptic if and only if $\text{trace } A \in (-1, 3)$. The region with $f(z) < 0$ is inside the deltoid shown in Figure 1.1 and that with $f(z) > 0$ is outside.

1.1.3 Geodesics

There are two types of totally geodesic submanifolds of real dimension two in $\mathbf{H}_{\mathbb{C}}^2$, *complex geodesics* (complex slices) and *real slices*. Complex geodesics are

acquired by the projection of two-dimensional complex subspaces of $\mathbb{C}^{2,1}$. Any complex geodesic is isometric to $\{[z : 0 : 1] | z \in \mathbb{C}\}$ in the projective model or to $\{(z, 0) | z \in \mathbb{C}\}$ in the ball model. Given any two points in $\mathbf{H}_{\mathbb{C}}^2$ there is a unique complex geodesic containing them. Any positive vector $c \in \mathbb{C}^{2,1}$ determines a two-dimensional complex subspace,

$$\{z \in \mathbb{C}^{2,1} | \langle c, z \rangle = 0\}.$$

Hence by projecting this subspace, it determines a complex geodesic. The vector c is called a *polar vector* of this geodesic. Conversely, any complex geodesic is represented by a positive vector. To describe possible mutual configurations of two complex geodesics we will need the following notion:

Definition. *The Hermitian cross-product is defined to be*

$$\begin{aligned} \boxtimes & : \mathbb{C}^{2,1} \times \mathbb{C}^{2,1} \rightarrow \mathbb{C}^{2,1} \\ z \boxtimes w & = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\bar{z} \times \bar{w}) = \begin{pmatrix} \overline{z_3 w_2} - \overline{z_2 w_3} \\ \overline{z_1 w_3} - \overline{z_3 w_1} \\ \overline{z_1 w_2} - \overline{z_2 w_1} \end{pmatrix}. \end{aligned}$$

Suppose we have two complex geodesics, C_1 and C_2 , with respective polar vectors, c_1 and c_2 . We can normalise our polar vectors so that

$$\langle c_1, c_1 \rangle = \langle c_2, c_2 \rangle = 1,$$

and assuming C_1 and C_2 do not coincide, we then have three possible configurations:

- the complex geodesics C_1 and C_2 intersect at a single point in $\mathbf{H}_{\mathbb{C}}^2$ if and only if

$$|\langle c_1, c_2 \rangle| < 1.$$

In which case $c_1 \boxtimes c_2$ is a negative vector corresponding to the intersection point of C_1 and C_2 and the angle of intersection $\angle(C_1, C_2) \in [0, \pi/2]$ can be defined by $\cos \angle(C_1, C_2) = |\langle c_1, c_2 \rangle|$.

- the complex geodesics C_1 and C_2 intersect at a single point in $\partial \mathbf{H}_{\mathbb{C}}^2$ if and only if

$$|\langle c_1, c_2 \rangle| = 1.$$

In which case $c_1 \boxtimes c_2$ is a null vector corresponding to the intersection point.

- the complex geodesics C_1 and C_2 are disjoint in $\mathbf{H}_{\mathbb{C}}^2 \cup \partial \mathbf{H}_{\mathbb{C}}^2$ if and only if

$$|\langle c_1, c_2 \rangle| > 1.$$

In which case the distance l between C_1 and C_2 is given by $\cosh(l/2) = |\langle c_1, c_2 \rangle|$.

Now a real slice is isometric to $\{[z : w : 1] \in \mathbf{H}_{\mathbb{C}}^2 | z, w \in \mathbb{R}\}$ in the projective model or $\{(z, w) | z, w \in \mathbb{R}\}$ in the ball model. A real slice in the ball model, where $\mathbb{R}^2 \subset \mathbb{C}^2$, is $PU(2, 1)$ equivalent to $\mathbb{R}^2 \cap \mathbf{H}_{\mathbb{C}}^2$.

Definition. *Given a complex geodesic C , there is a unique isometry $\iota_C \in PU(2, 1)$ of order 2, whose fixed point set is equal to C . This isometry is called the inversion on C (or complex reflection in C) and is given by*

$$\iota_C(z) = -z + 2 \frac{\langle z, c \rangle}{\langle c, c \rangle} c,$$

where c is a polar vector of C .

We can normalise this polar vector so that $\langle c, c \rangle = 1$, then we can simplify the above formula to

$$\iota_C(z) = -z + 2\langle z, c \rangle c.$$

Definition. Given a complex geodesic C and a unit complex number μ , a complex μ -reflection in C is a rotation around C by the angle $\arg(\mu)$ and is given by

$$t_C^\mu(z) = -z + (1 - \mu) \frac{\langle z, c \rangle}{\langle c, c \rangle} c,$$

where c is a polar vector of C .

Remark 1.1.3.1. If μ is an n -th root of unity, then the μ -reflection has order n . For $\mu = -1$ we get the usual complex reflections described above.

Note that a complex geodesic, in the ball model, is homeomorphic to a disc and its intersection with the boundary is homeomorphic to a circle.

Definition. Circles that arise as the boundaries of complex geodesics are called chains.

Remark 1.1.3.2. From two distinct points on a chain we can retrieve the complex geodesic through them, so there is a bijection between chains and complex geodesics. We can therefore, without loss of generality, talk about reflections in chains as well as reflections in complex geodesics.

We now discuss the representations of the chains in Heisenberg space, $\mathcal{H} = \mathbb{C} \times \mathbb{R} \cup \{\infty\}$, with points either ∞ or (ζ, v) with $\zeta \in \mathbb{C}$ and $v \in \mathbb{R}$. The most important thing we need to know about the \mathcal{H} -representation is the way in which chains appear. See [11, Section 4.3] for more details. Chains passing through ∞ are represented as vertical straight lines defined by $\zeta = \zeta_0$, such chains are called *vertical*. The vertical chain C_{ζ_0} defined by $\zeta = \zeta_0$ has a polar vector

$$c_{\zeta_0} = \begin{bmatrix} 1 \\ -\bar{\zeta}_0 \\ \bar{\zeta}_0 \end{bmatrix}. \quad (1.1.2)$$

Inversion in C_{ζ_0} is given by the matrix

$$\iota_{C_{\zeta_0}}(z) = \begin{pmatrix} 1 & -2\zeta & -2\zeta \\ -2\bar{\zeta} & 2|\zeta|^2 - 1 & 2|\zeta|^2 \\ 2\bar{\zeta} & -2|\zeta|^2 & -2|\zeta|^2 - 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

A chain not containing ∞ is called *finite*. A finite chain is represented by an ellipse whose vertical projection $\mathbb{C} \times \mathbb{R} \rightarrow \mathbb{C}$ is a (Euclidean) circle in \mathbb{C} . The finite chain with centre $(\zeta_0, v_0) \in \mathcal{H}$ and radius $r_0 > 0$ has a polar vector

$$\begin{bmatrix} 2\zeta_0 \\ 1 + r_0^2 - \zeta_0\bar{\zeta}_0 + iv_0 \\ 1 - r_0^2 + \zeta_0\bar{\zeta}_0 - iv_0 \end{bmatrix} \quad (1.1.3)$$

and consists of all $(\zeta, v) \in \mathcal{H}$ satisfying the equations

$$|\zeta - \zeta_0| = r_0, \quad v = v_0 - 2\text{Im}(\zeta\bar{\zeta}_0).$$

In particular the finite chain with centre $(0, 0)$ and radius 1 is the unit circle in the $\mathbb{C} \times \{0\}$ plane. It has a polar vector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and inversion in this chain is given by the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

1.2 Complex Hyperbolic Triangles

Definition. A complex hyperbolic triangle is a triple (C_1, C_2, C_3) of complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$.

The various types of triangle groups depend on how the complex geodesics C_k intersect each other. We will study two general types of complex hyperbolic triangles. The first kind is when each two complex geodesics intersect at a single point in $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$, that is all three vertices lie in $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$. We will assume that the complex geodesics C_{k-1} and C_{k+1} meet at an angle π/p_k , with $2 \leq p_k \in \mathbb{Q}$ and $1/p_1 + 1/p_2 + 1/p_3 < 1$, where $p_{k+3} = p_k$. If C_{k-1} and C_{k+1} are asymptotic then they intersect on the boundary with angle 0 and we write $p_k = \infty$.

Definition. A complex hyperbolic triangle (C_1, C_2, C_3) with vertices inside $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$ that meet at angles π/p_1 , π/p_2 and π/p_3 respectively is called a (p_1, p_2, p_3) triangle or alternatively we can call it a $(\varphi_1, \varphi_2, \varphi_3)$ triangle, where $\varphi_k = \pi/p_k$.

The second kind of triangle is when the complex geodesics do not intersect in $\mathbf{H}_{\mathbb{C}}^2$. In this case the complex geodesics C_{k-1} and C_{k+1} are ultra-parallel and we denote the distance l_k . If C_{k-1} and C_{k+1} are asymptotic they intersect on the boundary with distance 0 and we write $l_k = 0$.

Definition. A complex hyperbolic triangle (C_1, C_2, C_3) where the complex geodesics C_{k-1} and C_{k+1} do not meet pairwise inside $\mathbf{H}_{\mathbb{C}}^2$ but instead are distance l_k apart in $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$ is called an $[l_1, l_2, l_3]$ triangle.

Let the complex reflection ι_k denote an inversion on the corresponding geodesic C_k .

Definition. We call a subgroup of $PU(2, 1)$, generated by complex reflections ι_k for a complex hyperbolic (p_1, p_2, p_3) triangle, a (p_1, p_2, p_3) triangle group.

Note that this works for $p_1, p_2, p_3 \in \mathbb{Q}^+$. Consider the group,

$$\Gamma(p_1, p_2, p_3) = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_k^2 = (\gamma_{k-1}\gamma_{k+1})^{p_k} = 1 \quad \forall k \in \{1, 2, 3\} \rangle,$$

where $\gamma_{k+3} = \gamma_k$, which is isomorphic to the real hyperbolic triangle group of the triangle with angles π/p_k .

Definition. A representation of the group $\Gamma(p_1, p_2, p_3)$ into the group $PU(2, 1)$, given by taking the generators γ_k of $\Gamma(p_1, p_2, p_3)$ to the generators ι_k in $PU(2, 1)$ of a (p_1, p_2, p_3) -triangle group is a (p_1, p_2, p_3) -representation.

Abusing notation, we also say that Γ is a triangle group, and we say that ι_1 , ι_2 and ι_3 generate the triangle group Γ .

In the ultra-parallel case we define similarly $[l_1, l_2, l_3]$ triangle groups and $[l_1, l_2, l_3]$ representations. In each case the question may be asked of when a given triangle group is discrete. By discrete we mean that the image of the representation is discrete in the topology induced on $PU(2, 1)$ from $GL(3, \mathbb{C})$.

For each fixed triple (p_1, p_2, p_3) {resp. $[l_1, l_2, l_3]$ } in $\mathbf{H}_{\mathbb{C}}^2$, the space of non-isometric triangles with angles π/p_1 , π/p_2 and π/p_3 {resp. distances l_1 , l_2 and l_3 } is of real dimension one. This leads to a real one-dimensional family of non-conjugate triangle representations for a (p_1, p_2, p_3) {resp. $[l_1, l_2, l_3]$ } triangle group in $SU(2, 1)$.

Let c_k be the normalised polar vector of the complex geodesic C_k and denote $r_k = |\langle c_{k-1}, c_{k+1} \rangle|$. We have $r_k = \cos(\pi/p_k)$ and $r_k = \cosh(l_k/2)$ respectively. We can describe a parametrisation of the space of non-isometric complex hyperbolic triangles in $\mathbf{H}_{\mathbb{C}}^2$ by means of an angular invariant α .

Definition. We define the angular invariant α of the triangle (C_1, C_2, C_3) by

$$\alpha = \arg \left(\prod_{k=1}^3 \langle c_{k-1}, c_{k+1} \rangle \right).$$

Remark 1.2.0.3. α is invariant under isometries of $\mathbf{H}_{\mathbb{C}}^2$.

We have the following proposition, given in Anna Pratoussevitch's paper [27, Section 3, Prop.1], which gives us certain criteria for the cosine of the angular invariant.

Proposition 1.2.0.4. *A (p_1, p_2, p_3) {resp. $[l_1, l_2, l_3]$ } triangle in $\mathbf{H}_{\mathbb{C}}^2$ is determined uniquely, up to holomorphic isometry, by the triple (p_1, p_2, p_3) {resp. $[l_1, l_2, l_3]$ } and the angular invariant α , given above. For any $\alpha \in [0, 2\pi]$ there exists a (p_1, p_2, p_3) {resp. $[l_1, l_2, l_3]$ } triangle in $\mathbf{H}_{\mathbb{C}}^2$ with angular invariant α if and only if*

$$\cos \alpha < \frac{r_1^2 + r_2^2 + r_3^2 - 1}{2r_1 r_2 r_3},$$

where $r_k = \cos(\pi/p_k)$ and $r_k = \cosh(l_k/2)$ respectively.

Proof. [27, for proof, see section 3].

□

This inequality means that the triangle exists for values of parameter α in a symmetric open neighbourhood of π and does not exist for values outside of this neighbourhood.

Definition. *At a value of α such that the above inequality is not satisfied, we say that the triangle group is degenerate.*

Remark 1.2.0.5. *The triangle groups with angular invariant α and $2\pi - \alpha$ are conjugate via an anti-holomorphic isometry of $\mathbf{H}_{\mathbb{C}}^2$. Therefore we may restrict ourselves to the cases of angular invariant $\alpha \in [0, \pi]$.*

Remark 1.2.0.6. *For a (p_1, p_2, p_3) triangle, the angular invariant α with*

$$\cos \alpha = \frac{r_1^2 + r_2^2 + r_3^2 - 1}{2r_1 r_2 r_3}$$

corresponds to the case when all three complex geodesics meet at a point. In the case of angular invariant $\alpha = \pi$ all the vertices of the triangle lie in one real slice.

Remark 1.2.0.7. *A triangle group corresponding to a triangle with at least one vertex on the boundary exists for all $\alpha \in (0, 2\pi)$.*

Now obviously as we change α away from π we will always have the group relations, $\iota_k^2 = (\iota_{k-1}\iota_{k+1})^{p_k} = 1$ and their conjugates, but extra relations for longer products of elements may occur as the triangle group is deformed.

Chapter 2

Triangle Groups with Two Elliptic Elements

In this chapter we concentrate on the complex hyperbolic triangle group $\Gamma(p_1, p_2, p_3)$. We focus on the elements $w_A := \iota_3 \iota_2 \iota_3 \iota_1$ and $w_B := \iota_1 \iota_2 \iota_3$ and consider the case when both w_A and w_B are elliptic. If the triangle group Γ contains elliptic elements w_A and w_B then Γ is discrete only if w_A and w_B are of finite order. We use this fact together with trace formulas from [27] to reduce the question of discreteness to equation (2.1.3) which is a sum of seven cosines of rational multiples of π equal to 1. Namely Γ is discrete only if (2.1.3) has solutions of a specific kind.

In sections 2.2 and 2.3 we look at specific types of triangle groups that reduce equation (2.1.3) to a shorter sum of cosines so that we can use work by Conway and Jones [2] and Parker [21]. Conway and Jones [2] classified rational linear combinations of at most four cosines of rational multiples of π which are rational. Parker [21] classified rational linear combinations of at most six cosines of rational multiples of π which vanish. We use these results to find solutions to equation (2.1.3), which leads us to some non-discreteness results.

In section 2.4 we extend the results of Conway and Jones and Parker to longer

rational linear combinations of cosines of rational multiples of π , so that in theory we could solve equation (2.1.3) for any (p_1, p_2, p_3) triangle group.

2.1 Background of Triangle Groups

In view of Remark 1.2.0.5 we only consider $\alpha \in [0, \pi]$ but for some computations the parametrisation of complex hyperbolic triangle groups by

$$t = \left(\tan \frac{\alpha}{2} \right)^{-1} = \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}}$$

is more suitable than the parametrisation by the angular invariant α itself. Then a triangle group exists in the region $t \in [0, t_\infty)$, where

$$t_\infty = \sqrt{\frac{1 + c_\infty}{1 - c_\infty}} \quad \text{and} \quad c_\infty = \frac{r_1^2 + r_2^2 + r_3^2 - 1}{2r_1 r_2 r_3}.$$

For $c_\infty > 1$ we set $t_\infty = +\infty$.

When $t = 0$ all vertices lie in one real slice and hence the triangle group, which is a real hyperbolic triangle group, is a discrete subgroup of $PSL(2, \mathbb{R})$. As we change t we wish to find out whether the triangle group stays discrete. Maybe when we deform away from $t = 0$ we have extra relations between elements other than the known group relations. Intuitively the first products to inspect would be a product of three generators and then a product of four generators. So we consider the two elements

$$w_A := \iota_3 \iota_2 \iota_3 \iota_1 \quad \text{and} \quad w_B := \iota_1 \iota_2 \iota_3.$$

To simplify notation we just write $w_A = \iota_{3231}$ and $w_B = \iota_{123}$. Why do we choose $w_A = \iota_{3231}$ and $w_B = \iota_{123}$ in particular? Well, any product of the three distinct generators is conjugate to w_B and a product of four generators, which is not conjugate to a shorter product, is conjugate to either $w_A = \iota_{3231}$, ι_{1312} or ι_{2123} .

But Pratussevitch [27, Section 11] showed that w_A is the first of these to become elliptic as $t > 0$ tends to t_∞ .

Let t_A denote the point at which w_A is parabolic and similarly t_B denotes the first point when w_B is parabolic.

Definition. *If $t_A < t_B$, the element w_A becomes elliptic before w_B as $t > 0$ tends to t_∞ , we say that the triple (p_1, p_2, p_3) is of type A. Otherwise we say it is of type B.*

To check if an element is elliptic we use the trace of the element and Goldman's discriminant function (1.1.1). Pratussevitch [27, Section 8] gave the following formulas for the traces of w_A and w_B :

$$\begin{aligned}\text{trace}(w_A) &= 16r_1^2r_2^2 + 4r_3^2 - 1 - 16r_1r_2r_3 \cos \alpha \\ \text{trace}(w_B) &= 8r_1r_2r_3e^{i\alpha} - 4(r_1^2 + r_2^2 + r_3^2) + 3.\end{aligned}$$

Remark 2.1.0.8. *The trace of w_A is real.*

Proposition 2.1.0.9. *The equation in Proposition 1.2.0.4 is equivalent to*

$$\text{Re}(\text{tr}(w_B)) < -1.$$

As explained in section 1.1.2, an element $A \in SU(2, 1)$ with real trace is regular elliptic if and only if $\text{trace } A \in (-1, 3)$. Because of this property it is more straight forward to calculate when w_A is elliptic. From [27, Section 11, Prop. 12] we know that the element w_A is regular elliptic if and only if $|t| > t_A$, where

$$t_A = \sqrt{\frac{1 + c_A}{1 - c_A}} \quad \text{and} \quad c_A = \frac{4r_1^2r_2^2 + r_3^2 - 1}{4r_1r_2r_3}.$$

For $c_A > 1$ we set $t_A = \infty$. In general, to check the ellipticity of w_B is computationally much more involved. When $t = 0$ the triangle lies in a real slice and

so we know w_B is loxodromic. As we increase t , w_B may become parabolic then elliptic, then go back to being loxodromic then return to being elliptic and so on.

Lemma 2.1.0.10. *Suppose that A is an elliptic element of $SU(2, 1)$ with real trace. Then we may write $\text{trace } A = 1 + 2 \cos \theta$, for $\theta \in [0, 2\pi]$.*

Proof. Let $A \in SU(2, 1)$ be an elliptic element with real trace. Then the eigenvalues of A are $e^{i\theta}$, $e^{i\phi}$ and $e^{-i\theta-i\phi}$ for some θ and ϕ and

$$\begin{aligned} \text{Im}(\text{tr}(A)) = 0 &\Leftrightarrow \sin(\theta) + \sin(\phi) - \sin(\theta + \phi) = 0 \\ &\Leftrightarrow 4 \sin(\theta/2) \sin(\phi/2) \sin(\theta/2 + \phi/2) = 0. \end{aligned}$$

Therefore at least one of θ , ϕ or $\theta + \phi$ is an integer multiple of 2π and hence A has an eigenvalue 1. Thus the remaining two eigenvalues are conjugate and we have $\text{trace } A = 1 + 2 \cos \theta$. \square

Assume that w_A is elliptic of finite order n . Clearly the order of w_A must be larger than 1 if w_A is elliptic. Fixing the order of w_A does not necessarily specify a single point in the deformation space as there may be two or more values of t at which w_A has such an order. Using Lemma 2.1.0.10 we obtain $\text{trace}(w_A) = 1 + 2 \cos(2\pi m/n)$ for some m , where $\gcd(m, n) = 1$. We will only consider the cases when $m = 1$, in which case it is clear at what point of the deformation space we are.

Definition. *If we fix w_A to be a finite elliptic element of order n with $\text{trace}(w_A) = 1 + 2 \cos(2\pi/n)$, then this group is denoted by $\Gamma(p_1, p_2, p_3; n)$.*

Remark 2.1.0.11. *The case when $\text{trace}(w_A) = 1 + 2 \cos(2m\pi/n)$, with $m \neq 1$ is discussed by Thompson [33], these groups are called non-standard deformed triangle groups and denoted by $\Gamma(p_1, p_2, p_3; n/m)$.*

Assume that w_B is elliptic of finite order, then the eigenvalues of the matrix of w_B are roots of unity, say e^{ix} , e^{iy} and e^{iz} , where $x, y, z \in \mathbb{Q}\pi$. We can also assume

that the matrix of w_B has determinant equal to 1, which implies $e^{iz} = e^{-i(x+y)}$ and hence $\text{trace}(w_B) = e^{ix} + e^{iy} + e^{-i(x+y)}$.

As we touched on in the introduction, Goldman and Parker [10] were the first to study triangle groups for the whole deformation space of non-isometric complex hyperbolic triangles with fixed angles. They first looked at the ideal (∞, ∞, ∞) triangle groups. Given a complex hyperbolic ideal triangle group representation

$$\phi : \Gamma(\infty, \infty, \infty) \rightarrow PU(2, 1)$$

and the corresponding complex hyperbolic ideal triangle group $G := \phi(\Gamma(\infty, \infty, \infty))$, the question they were interested in was, ‘when is ϕ a discrete and faithful representation?’. Using the parameter $t \in [0, \infty)$ they showed that for $t \leq \sqrt{35}$, ϕ is a discrete embedding and that for $t > t_B = \sqrt{125/3}$, ϕ is no longer faithful because for $t > t_B$, w_B is elliptic and hence ϕ is not injective. They conjectured that in fact ϕ is a discrete embedding if and only if $t \leq t_B$. Then Schwartz [29] proved a stronger version of this conjecture, he proved that ϕ is discrete and faithful if and only if $t \leq t_B$ and furthermore that if $t > t_B$ then ϕ is not discrete.

In 2000, Wyss-Galifent [34] studied the $(4, 4, \infty)$ triangle groups and he found that there were discrete but not faithful representations of this type. These representations corresponded to values of t where w_A is elliptic and of finite order. Influenced by this result, Schwartz conjectured that complex hyperbolic triangle groups fall into two categories, those of type A and those of type B .

The conjectural picture, according to Schwartz [30] was as follows:

Conjecture 2.1.0.12. *We assume $3 \leq p_1 \leq p_2 \leq p_3$. A (p_1, p_2, p_3) -representation is a discrete embedding if and only if neither w_A nor w_B is elliptic. The set of corresponding parameter values is a closed symmetric interval. [The parameter referred to here is not the angular invariant α , but the parameter $t = (\tan \frac{\alpha}{2})^{-1}$ introduced at the beginning of this section and the symmetry refers to the symmetry*

about zero.]

If the triple (p_1, p_2, p_3) is of type A, then there is a countable collection of parameters for which the (p_1, p_2, p_3) -representation is infinite and discrete but not injective. If the triple (p_1, p_2, p_3) is of type B, then there are no such discrete but not injective (p_1, p_2, p_3) -representations.

In 2003, Schwartz [31] showed that the $(4, 4, 4; n)$ triangle groups are discrete for $n = 5, 6, 7, 8$ and 12. In fact in 2006, Deraux [6] proved that the $(4, 4, 4; 5)$ triangle group is a cocompact lattice by demonstrating that a Dirichlet domain for the group is bounded. Parker [23] studied all discrete triangle groups of type $(p, p, p; n)$ using the result of Conway and Jones [2] and found counterexamples to Schwartz's conjecture above, one such counterexample was the $(18, 18, 18; 18)$ triangle group.

Pratoussevitch [28] showed that (m, m, ∞) triangle groups are not discrete if the product of three generators w_B is regular elliptic and Kamiya, Parker and Thompson [16] have used this result to show that a large number of (m, m, ∞) triangle groups are non-discrete.

We consider the case when both w_A and w_B are elliptic. If the triangle group is discrete then w_A and w_B must be elliptic of finite order.

Now using the formulas for $\text{trace}(w_A)$ and $\text{trace}(w_B)$, we obtain:

$$1 + 2 \cos\left(\frac{2\pi}{n}\right) = 16r_1^2 r_2^2 + 4r_3^2 - 1 - 16r_1 r_2 r_3 \cos \alpha \quad (2.1.1)$$

$$e^{ix} + e^{iy} + e^{iz} = 8r_1 r_2 r_3 e^{i\alpha} - 4(r_1^2 + r_2^2 + r_3^2) + 3. \quad (2.1.2)$$

We can rearrange equation (2.1.1) in terms of $\cos \alpha$, and take the real part of equation (2.1.2) and then rearrange in terms of $\cos \alpha$. So we have the following

two equations:

$$\begin{aligned}\cos \alpha &= \frac{8r_1^2 r_2^2 + 2r_3^2 - \cos\left(\frac{2\pi}{n}\right) - 1}{8r_1 r_2 r_3}, \\ \cos \alpha &= \frac{\cos x + \cos y + \cos z + 4(r_1^2 + r_2^2 + r_3^2) - 3}{8r_1 r_2 r_3}.\end{aligned}$$

Now equating the above for $\cos \alpha$ we find:

$$\begin{aligned}8r_1^2 r_2^2 + 2r_3^2 - \cos\left(\frac{2\pi}{n}\right) - 1 &= \cos x + \cos y + \cos z + 4(r_1^2 + r_2^2 + r_3^2) - 3 \\ \Leftrightarrow 8r_1^2 r_2^2 + 2r_3^2 - \cos\left(\frac{2\pi}{n}\right) - \cos x - \cos y - \cos z - 4(r_1^2 + r_2^2 + r_3^2) + 2 &= 0 \\ \Leftrightarrow 8r_1^2 r_2^2 - 4r_1^2 - 4r_2^2 - 2r_3^2 - \cos\left(\frac{2\pi}{n}\right) - \cos x - \cos y - \cos z + 2 &= 0.\end{aligned}$$

If we now use that $2r_k^2 = \cos(2\varphi_k) + 1$, where $\varphi_k = \pi/p_k$, our equation becomes:

$$\begin{aligned}2(\cos(2\varphi_1) + 1)(\cos(2\varphi_2) + 1) - 2(\cos(2\varphi_1) + 1) - 2(\cos(2\varphi_2) + 1) \\ - (\cos(2\varphi_3) + 1) - \cos\left(\frac{2\pi}{n}\right) - \cos x - \cos y - \cos z + 2 &= 0 \\ \Leftrightarrow 2 \cos(2\varphi_1) \cos(2\varphi_2) - \cos(2\varphi_3) - \cos\left(\frac{2\pi}{n}\right) - \cos x - \cos y - \cos z &= 1.\end{aligned}$$

Finally using the identity $2 \cos(2\varphi_1) \cos(2\varphi_2) = \cos(2\varphi_1 + 2\varphi_2) + \cos(2\varphi_1 - 2\varphi_2)$ we obtain an equation given as a rational linear combination of cosines of rational multiples of π :

$$\begin{aligned}1 &= \cos(2\varphi_1 + 2\varphi_2) + \cos(2\varphi_1 - 2\varphi_2) - \cos(2\varphi_3) \\ &\quad - \cos x - \cos y - \cos z - \cos \frac{2\pi}{n}.\end{aligned}\tag{2.1.3}$$

Remark 2.1.0.13. *Existence of $x, y, z \in \mathbb{Q}\pi$ and $n \in \mathbb{Z}^+$ satisfying equation (2.1.3) is necessary but not sufficient for discreteness.*

If we find suitable $x, y \in \mathbb{Q}\pi$ and $n \in \mathbb{Z}^+$ that satisfy equation (2.1.3) then we need also check whether the imaginary part of equation (2.1.2) is also satisfied.

That is

$$\sin x + \sin y + \sin z = 8r_1r_2r_3 \sin \alpha. \quad (2.1.4)$$

Remark 2.1.0.14. *But again an important remark is that the existence of $x, y, z \in \mathbb{Q}\pi$ and $n \in \mathbb{Z}^+$ satisfying equation (2.1.3) and also satisfying (2.1.4) is necessary but not sufficient for discreteness.*

Now equation (2.1.3) is given as a rational linear combination of cosines of rational multiples of π equal to one. Conway and Jones [2] classified rational linear combinations of at most four cosines of rational multiples of π which is rational. Parker [21] classified rational linear combinations of at most six cosines of rational multiples of π which vanish. We will now state the Theorem of Parker.

Theorem 2.1.0.15 (Theorem A.1.1 [21]). *Suppose we have at most six distinct rational multiples of π lying strictly between 0 and $\pi/2$, for which some rational linear combination of their cosines is zero but no proper subset has this property, then the appropriate linear combination is proportional to one of the following:*

$$0 = \sum_{k=0}^2 \cos\left(\varphi + \frac{2k\pi}{3}\right), \quad \varphi \in (0, \pi), \quad \varphi \neq \frac{m\pi}{6} \quad (2.1.5)$$

$$0 = \sum_{k=0}^4 \cos\left(\varphi + \frac{2k\pi}{5}\right), \quad \varphi \in (0, \pi), \quad \varphi \neq \frac{n\pi}{10} \quad (2.1.6)$$

$$0 = \sum_{k=1}^2 \cos\left(\varphi + \frac{2k\pi}{3}\right) - \sum_{k=1}^4 \cos\left(\varphi + \frac{2k\pi}{5}\right), \quad \varphi \in (0, \pi), \quad \varphi \neq \frac{m\pi}{6}, \frac{n\pi}{10} \quad (2.1.7)$$

$$0 = \cos \frac{\pi}{3} - \cos \frac{\pi}{5} + \cos \frac{2\pi}{5} \quad (2.1.8)$$

$$0 = \cos \frac{\pi}{3} - \cos \frac{\pi}{7} + \cos \frac{2\pi}{7} - \cos \frac{3\pi}{7} \quad (2.1.9)$$

$$0 = \cos \frac{\pi}{3} - \cos \frac{\pi}{11} + \cos \frac{2\pi}{11} - \cos \frac{3\pi}{11} + \cos \frac{4\pi}{11} - \cos \frac{5\pi}{11} \quad (2.1.10)$$

$$0 = \cos \frac{\pi}{3} - \cos \frac{\pi}{5} + \cos \frac{\pi}{15} - \cos \frac{4\pi}{15} \quad (2.1.11)$$

$$0 = \cos \frac{\pi}{3} + \cos \frac{2\pi}{5} - \cos \frac{2\pi}{15} + \cos \frac{7\pi}{15} \quad (2.1.12)$$

$$0 = \cos \frac{\pi}{3} - \cos \frac{\pi}{7} + \cos \frac{2\pi}{7} - \cos \frac{2\pi}{21} + \cos \frac{5\pi}{21} \quad (2.1.13)$$

$$0 = \cos \frac{\pi}{3} - \cos \frac{\pi}{7} - \cos \frac{3\pi}{7} + \cos \frac{\pi}{21} - \cos \frac{8\pi}{21} \quad (2.1.14)$$

$$0 = \cos \frac{\pi}{3} + \cos \frac{2\pi}{7} - \cos \frac{3\pi}{7} - \cos \frac{4\pi}{21} - \cos \frac{10\pi}{21} \quad (2.1.15)$$

$$0 = \cos \frac{\pi}{3} - \cos \frac{\pi}{7} + \cos \frac{\pi}{21} - \cos \frac{2\pi}{21} + \cos \frac{5\pi}{21} - \cos \frac{8\pi}{21} \quad (2.1.16)$$

$$0 = \cos \frac{\pi}{3} + \cos \frac{2\pi}{7} - \cos \frac{2\pi}{21} - \cos \frac{4\pi}{21} + \cos \frac{5\pi}{21} - \cos \frac{10\pi}{21} \quad (2.1.17)$$

$$0 = \cos \frac{\pi}{3} - \cos \frac{3\pi}{7} + \cos \frac{\pi}{21} - \cos \frac{4\pi}{21} - \cos \frac{8\pi}{21} - \cos \frac{10\pi}{21} \quad (2.1.18)$$

$$0 = \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} - \cos \frac{\pi}{7} + \cos \frac{2\pi}{7} - \cos \frac{3\pi}{7} \quad (2.1.19)$$

$$0 = \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} - \cos \frac{\pi}{7} + \cos \frac{2\pi}{7} - \cos \frac{2\pi}{21} + \cos \frac{5\pi}{21} \quad (2.1.20)$$

$$0 = \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} - \cos \frac{\pi}{7} - \cos \frac{3\pi}{7} + \cos \frac{\pi}{21} - \cos \frac{8\pi}{21} \quad (2.1.21)$$

$$0 = \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} + \cos \frac{2\pi}{7} - \cos \frac{3\pi}{7} - \cos \frac{4\pi}{21} - \cos \frac{10\pi}{21} \quad (2.1.22)$$

We use these sums to give some non-discreteness results for certain types of triangle groups. One already known method used to show that a triangle group is non discrete is by using the Jørgensen inequality. Jørgensen inequality gives a necessary condition for a non-elementary two generator group of isometries of the real hyperbolic space to be discrete. This result has been generalised for complex hyperbolic triangle groups. We state and use the result of Corollary 2.4 of [15].

Lemma 2.1.0.16 (Jørgensen's Inequality). *Let $\Gamma(p_1, p_2, p_3)$ be a triangle group as defined above with $p_1 \leq p_2 \leq p_3$. Then if $p_1 \geq 7$ and*

$$(2\operatorname{Re}(tr(\iota_{123})) + tr(\iota_{23}) - 1)^2 < 3 - tr(\iota_{23})$$

then Γ is not discrete. If $p_2 \geq 7$ and

$$(2\operatorname{Re}(tr(\iota_{123})) + tr(\iota_{31}) - 1)^2 < 3 - tr(\iota_{31})$$

then Γ is not discrete. If $p_3 \geq 7$ and

$$(2\operatorname{Re}(tr(\iota_{123})) + tr(\iota_{12}) - 1)^2 < 3 - tr(\iota_{12})$$

then Γ is not discrete.

Remark 2.1.0.17. *These inequalities are the best possible, in the sense that there are discrete groups where we get equality, namely $(18, 18, 18; 18)$ and $(7, 7, 14; 4)$ triangle groups.*

In order to use the Jørgensen inequality it is necessary to know trace of $\iota_{k-1}\iota_{k+1}$. By a simple calculation or see [27, section 7] we have that

$$tr(\iota_{k-1}\iota_{k+1}) = tr(\iota_{k+1}\iota_{k-1}) = 4r_k^2 - 1.$$

Substituting the above and the formula given by Pratoševitch for the trace of

w_B into

$$(2\operatorname{Re}(tr(\iota_{123})) + tr(\iota_{k+1}\iota_{k-1}) - 1)^2 < 3 - tr(\iota_{k+1}\iota_{k-1}),$$

we may rewrite the inequality so that the the group is non-discrete for $p_k \geq 7$ provided

$$(16r_1r_2r_3 \cos \alpha - 8r_{k-1}^2 - 8r_k^2 - 8r_{k+1}^2 + 6 + 4r_k^2 - 1 - 1)^2 < 3 - 4r_k^2 + 1.$$

From Proposition 2.1.0.9 we have $2\operatorname{Re}(tr(\iota_{123})) < -2$ and using that $4r_k^2 - 2 \leq 2$ the term inside the brackets is negative. Hence, we can simplify to

$$-8r_1r_2r_3 \cos \alpha + 4r_{k-1}^2 + 2r_k^2 + 4r_{k+1}^2 - 2 < \sqrt{1 - r_k^2}.$$

Using equation (2.1.1) and rearranging gives

$$\sqrt{1 - r_k^2} > \cos(2\pi/n) - 1 - 8r_1^2r_2^2 - 2r_3^2 + 4r_{k-1}^2 + 2r_k^2 + 4r_{k+1}^2.$$

Therefore we may rewrite the three formulae in Lemma 2.1.0.16 as

$$\begin{aligned} \sqrt{1 - r_1^2} &> \cos(2\pi/n) - 1 - 8r_1^2r_2^2 + 2r_1^2 + 4r_2^2 + 2r_3^2, \\ \sqrt{1 - r_2^2} &> \cos(2\pi/n) - 1 - 8r_1^2r_2^2 + 4r_1^2 + 2r_2^2 + 2r_3^2, \\ \sqrt{1 - r_3^2} &> \cos(2\pi/n) - 1 - 8r_1^2r_2^2 + 4r_1^2 + 4r_2^2, \end{aligned}$$

respectively. We use the Jørgensen inequality to compare with our own results of non-discreteness, for example see Table 2.1. It is also shown in [15] that the triangle group (p_1, p_2, p_3) with $p_1 \leq p_2 \leq p_3 < \infty$ is not discrete if $p_1 > 31$.

Remark 2.1.0.18. *We can express Proposition 1.2.0.4 in terms of $\cos \frac{2\pi}{n}$ by substituting equation (2.1.1) into the inequality in Proposition 1.2.0.4. Thus there*

exists a complex hyperbolic triangle group $\Gamma(p_1, p_2, p_3; n)$ if and only if

$$\cos \frac{2\pi}{n} > 8r_1^2 r_2^2 - 4r_1^2 - 4r_2^2 - 2r_3^2 + 3.$$

2.2 (p, p, 2p) Case

In this section we look at the triangle groups with $(p_1, p_2, p_3) = (p, p, 2p)$ and prove some non-discreteness results. Using the trace formulas and the discriminant function f described in section 1.1.2, we can calculate (using MAPLE) that the $(p, p, 2p)$ triangle group, with $p \in \mathbb{Z}$, is of type B if $p \geq 13$. In the $(p, p, 2p)$ case equation (2.1.3) becomes

$$0 = \cos \frac{\pi}{p} - \cos \frac{4\pi}{p} + \cos x + \cos y + \cos z + \cos \frac{2\pi}{n}, \quad (2.2.1)$$

which is given as a linear combination of cosines of six rational multiples of π . If equation (2.2.1) is not satisfied for some $x, y, z \in \mathbb{Q}\pi$ and $n \in \mathbb{Z}^+$ then the group is not discrete. We can rewrite equation (2.2.1) as

$$\begin{aligned} \cos \frac{\pi}{p} - \cos \frac{4\pi}{p} + \cos \frac{2\pi}{n} &= -(\cos x + \cos y + \cos z) \\ &= -\operatorname{Re}(tr(w_B)). \end{aligned}$$

According to Proposition 2.1.0.9, a triangle exists if and only if $\operatorname{Re}(tr(w_B)) < -1$.

Hence,

$$\cos \frac{\pi}{p} - \cos \frac{4\pi}{p} + \cos \frac{2\pi}{n} > 1,$$

Also note that

$$-\cos \frac{4\pi}{p} + \cos \frac{2\pi}{n} > 1 - \cos \frac{\pi}{p} > 0,$$

and therefore require $p < 2n$.

Theorem 2.2.0.19. *The complex hyperbolic triangle group $\Gamma(p, p, 2p; n)$ is non-discrete for $p \in \mathbb{Z}$, when the $(p, p, 2p)$ triangle is of type B and w_B is regular elliptic.*

Remark 2.2.0.20. *The fact we consider the triangle group $\Gamma(p, p, 2p; n)$ with n finite implies that w_A is elliptic.*

Proof. We look for solutions to

$$0 = \cos \frac{\pi}{p} - \cos \frac{4\pi}{p} + \cos x + \cos y + \cos z + \cos \frac{2\pi}{n},$$

where $p \geq 13$, $n \geq 7$ and $x + y + z = 2k\pi$, $k \in \mathbb{Z}$. We can now use Theorem 2.1.0.15 but note that equation (2.2.1) can have subsets which sum to zero, angles that are not necessarily distinct and angles that do not necessarily lie in $(0, \pi/2)$. As $p \geq 13$ and $n \geq 7$, the only angles which could be outside the region $(0, \pi/2)$ can be x , y or z . Consider the case when one of the angles is $2k\pi$ for some $k \in \mathbb{Z}$, then without loss of generality we may assume $x = 2k\pi$. This then implies that

$$\text{tr}(w_B) = 1 + 2 \cos y \in (-1, 3),$$

which is a contradiction as $\text{tr}(w_B) < -1$. Similary if we assume $x = \pi + 2k\pi$ then $\text{tr}(w_B) = -1$, which is not possible. Therefore when looking for solutions to (2.2.1) we cannot use the angle $k\pi$.

If we have coinciding angles then we may rewrite equation (2.2.1) in the form

$$0 = A_1 \cos a_1\pi + A_2 \cos a_2\pi + A_3 \cos a_3\pi + \dots$$

where $A_i \in \mathbb{Z}$, $a_i \in \mathbb{Q}$ with some $A_i > 1$. From the list of sums in Theorem 2.1.0.15 we have no minimal sum of this form so we must therefore combine two smaller sums. Suppose equation (2.2.1) is not minimal then we may split it into

two or more smaller sums. The largest length of these smaller sums we call the maximum sum. Suppose the maximum sum is one i.e. each cosine vanishes; we know this is not possible as, for example $\frac{\pi}{p} \in (0, \frac{\pi}{3})$.

Suppose the maximum sum is two, that is one or more pair of cosines cancel. We have two possibilities, either we have one sum of length two and the rest of length one or we have three pairs of cosines which vanish. Now as

$$\cos \frac{\pi}{p} - \cos \frac{4\pi}{p} + \cos \frac{2\pi}{n} > 1,$$

none of these cosines can vanish as a pair and none vanish alone. So it must be that we have three pairs of cosines that vanish and we can write them in the form

$$\begin{aligned} \cos \frac{\pi}{p} + \cos \left(\pi - \frac{\pi}{p} \right) &= 0, \\ -\cos \frac{4\pi}{p} + \cos \frac{4\pi}{p} &= 0, \\ \cos \frac{2\pi}{n} + \cos \left(\pi - \frac{2\pi}{n} \right) &= 0. \end{aligned}$$

Now we must check whether one of $\frac{4\pi}{p}$, $\left(\pi - \frac{\pi}{p} \right)$ and $\left(\pi - \frac{2\pi}{n} \right)$ can be written as x , y and z , but we must also consider angles with the same cosine. In other words, we search for solutions to

$$\frac{4\pi}{p} \pm \frac{\pi}{p} \pm \frac{2\pi}{n} = 2k\pi,$$

where $k \in \mathbb{Z}$. We have four possibilities and we can check each one individually. First we consider

$$(++) : \quad \frac{5\pi}{p} + \frac{2\pi}{n} = 2k\pi,$$

but as $n > p/2$ and $p > 13$ we have that

$$0 < \frac{5\pi}{p} + \frac{2\pi}{n} < \frac{9\pi}{p} \leq \frac{9\pi}{13},$$

we have no solutions. Next we have

$$(+ -) : \quad \frac{5\pi}{p} - \frac{2\pi}{n} = 2k\pi,$$

but as

$$\frac{\pi}{p} < \frac{5\pi}{p} - \frac{2\pi}{n} < \frac{5\pi}{13},$$

we have no solutions. Next we consider

$$(- +) : \quad \frac{3\pi}{p} + \frac{2\pi}{n} = 2k\pi,$$

but as

$$0 < \frac{3\pi}{p} + \frac{2\pi}{n} < \frac{7\pi}{p} \leq \frac{7\pi}{13},$$

we have no solutions. Finally we have

$$(- -) : \quad \frac{3\pi}{p} - \frac{2\pi}{n} = 2k\pi,$$

but this time

$$-\frac{\pi}{p} < \frac{3\pi}{p} - \frac{2\pi}{n} < \frac{3\pi}{13},$$

so we have a solution when

$$\frac{3\pi}{p} - \frac{2\pi}{n} = 0 \Rightarrow n = \frac{2p}{3}.$$

For n to be an integer p must be a multiple of 3, therefore the following triangle groups may be discrete:

$$(15, 15, 30; 10)$$

$$(18, 18, 36; 12)$$

$$(21, 21, 42; 14)$$

$$(24, 24, 48; 16)$$

$$(27, 27, 54; 18)$$

$$(30, 30, 60; 20)$$

For $p > 31$ we know that the triangle group is not discrete, see [15]. We further investigate these triangle groups later in the proof.

Now we can systematically run through each sum in Theorem 2.1.0.15, looking for solutions to equation (2.2.1).

Sum (2.1.5) used twice: If we try and use (2.1.5) twice for a solution, then as $\frac{\pi}{p}, \frac{2\pi}{n} \in (0, \pi/3)$ and their cosines enter with the same sign, it is not possible for them to be in the same equation of the form (2.1.5). This leads to two sums equal to zero of the form,

$$\begin{aligned} 0 &= \cos \frac{\pi}{p} + \cos \left(\frac{\pi}{p} + \frac{2\pi}{3} \right) + \cos \left(\frac{\pi}{p} - \frac{2\pi}{3} \right), \\ 0 &= \cos \frac{2\pi}{n} + \cos \left(\frac{2\pi}{n} + \frac{2\pi}{3} \right) + \cos \left(\frac{2\pi}{n} - \frac{2\pi}{3} \right). \end{aligned}$$

We have two choices for which equation $-\cos \frac{4\pi}{p} = \cos \frac{(p-4)\pi}{p}$ belongs to.

i) First assume $\cos \frac{(p-4)\pi}{p}$ is given in the same equation as $\cos \frac{\pi}{p}$. Note that as $p \geq 13$ then $\frac{4\pi}{p} \in (0, \frac{\pi}{3})$ and subsequently

$$\pi - \frac{4\pi}{p} \in \left(\frac{2\pi}{3}, \pi \right) \Rightarrow \cos \left(\pi - \frac{4\pi}{p} \right) \in \left(-1, -\frac{1}{2} \right).$$

On the other hand,

$$\begin{aligned}\frac{\pi}{p} + \frac{2\pi}{3} \in \left(\frac{2\pi}{3}, \pi\right) &\Rightarrow \cos\left(\frac{\pi}{p} + \frac{2\pi}{3}\right) \in \left(-1, -\frac{1}{2}\right), \\ \frac{\pi}{p} - \frac{2\pi}{3} \in \left(-\frac{2\pi}{3}, -\frac{\pi}{3}\right) &\Rightarrow \cos\left(\frac{\pi}{p} - \frac{2\pi}{3}\right) \in \left(-\frac{1}{2}, \frac{1}{2}\right).\end{aligned}$$

Hence the only option is

$$\cos\left(\pi - \frac{4\pi}{p}\right) = \cos\left(\frac{\pi}{p} + \frac{2\pi}{3}\right).$$

Notice that both angles are in $\left(\frac{2\pi}{3}, \pi\right)$ and have the same cosine, hence

$$\pi - \frac{4\pi}{p} = \frac{\pi}{p} + \frac{2\pi}{3},$$

which implies $p = 15$ and then we have the following two equations,

$$\begin{aligned}\cos \frac{\pi}{15} + \cos \frac{11\pi}{15} + \cos \frac{3\pi}{5} &= 0 \\ \text{and} \quad \cos \frac{2\pi}{n} + \cos \left(\frac{2\pi}{n} + \frac{2\pi}{3}\right) + \cos \left(\frac{2\pi}{n} - \frac{2\pi}{3}\right) &= 0\end{aligned}$$

Now we must check whether one of $\frac{3\pi}{5}$, $\left(\frac{2\pi}{n} + \frac{2\pi}{3}\right)$ and $\left(\frac{2\pi}{n} - \frac{2\pi}{3}\right)$ can be written as x , y and z , but we must also consider angles with the same cosine. In other words, we search for solutions to

$$\frac{3\pi}{5} \pm \left(\frac{2\pi}{n} + \frac{2\pi}{3}\right) \pm \left(\frac{2\pi}{n} - \frac{2\pi}{3}\right) = 2k\pi,$$

where $k \in \mathbb{Z}$. Considering each of the four possible sign choices it is easily checked that no such suitable solution exists for n .

ii) Next assume $\frac{(p-4)\pi}{p}$ is given in the same equation as $\frac{2\pi}{n}$. Note that as

$n \geq 7$ then $\frac{2\pi}{n} \in (0, \frac{\pi}{3})$ and subsequently

$$\begin{aligned} \left(\frac{2\pi}{n} + \frac{2\pi}{3}\right) &\in \left(\frac{2\pi}{3}, \pi\right) \Rightarrow \cos\left(\frac{2\pi}{n} + \frac{2\pi}{3}\right) \in \left(-1, -\frac{1}{2}\right), \\ \left(\frac{2\pi}{n} - \frac{2\pi}{3}\right) &\in \left(-\frac{2\pi}{3}, -\frac{\pi}{3}\right) \Rightarrow \cos\left(\frac{2\pi}{n} - \frac{2\pi}{3}\right) \in \left(-\frac{1}{2}, \frac{1}{2}\right). \end{aligned}$$

On the other hand $p \geq 13$ so as before

$$\pi - \frac{4\pi}{p} \in \left(\frac{2\pi}{3}, \pi\right) \Rightarrow \cos\left(\pi - \frac{4\pi}{p}\right) \in \left(-1, -\frac{1}{2}\right).$$

Therefore the only option is

$$\cos \frac{(p-4)\pi}{p} = \cos\left(\frac{2\pi}{n} + \frac{2\pi}{3}\right).$$

Notice that both angles $\frac{(p-4)\pi}{p}$ and $\left(\frac{2\pi}{n} + \frac{2\pi}{3}\right)$ are in the interval $\left(\frac{2\pi}{3}, \pi\right)$ and their cosines coincide, hence

$$\begin{aligned} \frac{(p-4)\pi}{p} &= \frac{2\pi}{n} + \frac{2\pi}{3} \\ \Leftrightarrow p &= \frac{12n}{n-6}. \end{aligned}$$

In that case we are left with the following two sums

$$\begin{aligned} \cos \frac{\pi}{p} + \cos\left(\frac{\pi}{p} + \frac{2\pi}{3}\right) + \cos\left(\frac{\pi}{p} - \frac{2\pi}{3}\right) &= 0 \\ \text{and} \quad \cos \frac{2\pi}{n} + \cos\left(\frac{(p-4)\pi}{p}\right) + \cos\left(\frac{4\pi}{p} + \frac{\pi}{3}\right) &= 0 \end{aligned}$$

We must now check whether one of $\left(\frac{\pi}{p} + \frac{2\pi}{3}\right)$, $\left(\frac{\pi}{p} - \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{p} + \frac{\pi}{3}\right)$ can be written as x , y and z . Or equivalently we search for solutions to

$$\left(\frac{4\pi}{p} + \frac{\pi}{3}\right) \pm \left(\frac{\pi}{p} + \frac{2\pi}{3}\right) \pm \left(\frac{\pi}{p} - \frac{2\pi}{3}\right) = 2k\pi,$$

where $k \in \mathbb{Z}$. It is easily checked that there are no such solutions for a suitable $p \geq 13$.

Sum (2.1.5) used exactly once: If we use (2.1.5) once, then we have two equations which sum to zero and can be written in the form

$$0 = \cos \varphi + \cos \left(\varphi + \frac{2\pi}{3} \right) + \cos \left(\varphi - \frac{2\pi}{3} \right) = \cos a + \cos b + \cos c,$$

where $a, b, c \in \mathbb{Q}\pi$. As previously both $\cos \frac{\pi}{p}$ and $\cos \frac{2\pi}{n}$ cannot be in the same equation of the form (2.1.5). Suppose that both $\cos \frac{\pi}{p}$ and $\cos \frac{2\pi}{n}$ are not in (2.1.5) then we have

$$\cos \frac{\pi}{p} + \cos \frac{2\pi}{n} + \cos c = 0.$$

As $\cos \frac{\pi}{p}$, $p \geq 13$ cannot be written as an cosine from $\pm(2.1.8)$ hence the sum is not minimal. Therefore at least one of the cosines is 0 but $\cos \frac{\pi}{p}$, $\cos \frac{2\pi}{n} \neq 0$ so it must be that $\cos c = 0$. This then implies that

$$\cos \frac{\pi}{p} + \cos \frac{2\pi}{n} = 0,$$

which is a contradiction, as we need $\operatorname{Re}(tr(w_B)) < -1$ for the triangle group to exist. So it must be that $\cos \frac{\pi}{p}$ and $\cos \frac{2\pi}{n}$ belong to separate equations.

Similarly if we assume $\cos \frac{(p-4)\pi}{p}$ is not in (2.1.5) then we have the equation

$$\cos \frac{(p-4)\pi}{p} + \cos b + \cos c = 0,$$

where b is either $\frac{\pi}{p}$ or $\frac{2\pi}{n}$. Once more the sum is not minimal and it would again lead to $\operatorname{Re}(tr(w_B)) \geq -1$, which is not possible. Therefore we are left with two possibilities, either $\cos \frac{\pi}{p}$ and $\cos \frac{(p-4)\pi}{p}$ are given in (2.1.5) or $\cos \frac{2\pi}{n}$ and $\cos \frac{(p-4)\pi}{p}$ are given in (2.1.5).

- i) First assume $\cos \frac{\pi}{p}$ and $\cos \frac{(p-4)\pi}{p}$ are given by (2.1.5). As before, it must be that $p = 15$ and we have the following two equations

$$\begin{aligned} \cos \frac{\pi}{15} + \cos \frac{11\pi}{15} + \cos \frac{3\pi}{5} &= 0 \\ \text{and} \quad \cos \frac{2\pi}{n} + \cos b + \cos c &= 0. \end{aligned}$$

We need to match b, c and $\frac{3\pi}{5}$ with x, y and z . Assume that the second equation is minimal, therefore we must use sum $-(2.1.8)$ so that we have a cosine of the form $\cos \frac{2\pi}{n}$. That is, we have

$$0 = -\cos \frac{\pi}{3} + \cos \frac{\pi}{5} - \cos \frac{2\pi}{5},$$

where $\frac{2\pi}{n} = \frac{\pi}{5}$. The remaining three cosines must be $\cos x, \cos y$ and $\cos z$ where $x + y + z = 2k\pi, k \in \mathbb{Z}$. If we check the four sign combinations of

$$\frac{3\pi}{5} \pm \frac{3\pi}{5} \pm \frac{2\pi}{3} = 2k\pi,$$

we find no solutions. Now if the second equation is not minimal then it must be that one of the cosines is zero, but since $\cos \frac{2\pi}{n}$ does not vanish, we may therefore assume $\cos c = 0$ without a loss of generality. That is $c = \pm\pi/2 + 2k\pi$, where $k \in \mathbb{Z}$. Hence if we want the sum to be zero it must be that $b = \pm\left(\pi - \frac{2\pi}{n}\right) + 2l\pi$, where $l \in \mathbb{Z}$. Therefore to check whether the angles can be written in the form x, y and z we search for solutions to

$$\left(\pi - \frac{2\pi}{n}\right) \pm \frac{3\pi}{5} \pm \frac{\pi}{2} = 2k\pi,$$

where $k \in \mathbb{Z}$. This time we do find a suitable solution for n , namely when we take $(++)$ we obtain $n = 20$ as a possible solution. This

means the $(15, 15, 30; 20)$ triangle group may be discrete. This needs further inspection.

- ii) Next assume $\cos \frac{2\pi}{n}$ and $\cos \frac{(p-4)\pi}{p}$ are given by (2.1.5). As before it must be that

$$p = \frac{12n}{n-6}.$$

This will give us the following equations,

$$\begin{aligned} \cos \frac{2\pi}{n} + \cos \frac{(p-4)\pi}{p} + \cos \left(\frac{4\pi}{p} + \frac{\pi}{3} \right) &= 0 \\ \cos \frac{\pi}{p} + \cos b + \cos c &= 0. \end{aligned}$$

We need to match b, c and $\left(\frac{4\pi}{p} + \frac{\pi}{3} \right)$ with x, y and z . Now as $\cos \frac{\pi}{p}$ is not of the form of any of the cosines in (2.1.8). Therefore it must be that the sum is not minimal and hence one of the cosines is zero, but since $\cos \frac{\pi}{p}$ does not vanish, we may again assume $\cos c = 0$ without a loss of generality. That is $c = \pm\pi/2 + 2k\pi$, where $k \in \mathbb{Z}$. Hence if we want the sum to be zero it must be that $b = \pm \left(\pi - \frac{\pi}{p} \right) + 2l\pi$, where $l \in \mathbb{Z}$. Therefore to check whether the angles can be written in the form x, y and z we search for solutions to

$$\left(\frac{4\pi}{p} + \frac{\pi}{3} \right) \pm \left(\pi - \frac{\pi}{p} \right) \pm \frac{\pi}{2} = 2k\pi,$$

where $k \in \mathbb{Z}$. We find suitable solutions for p when we take $(++)$ and $(-+)$. For $(++)$ we obtain $p = 18$ as a possible solution, which implies $n = 18$. When we take $(-+)$ we get $p = 30$ with $n = 10$ but this is not a possible solution as we require $p < 2n$. Thus for $p = 18$ there exist an $n \in \mathbb{Z}$ and $x, y, z \in \mathbb{Q}\pi$ such that equation (2.2.1) is satisfied. Therefore the triangle groups $(18, 18, 36; 18)$ may

be discrete. But the $(18, 18, 36; 18)$ triangle group is isomorphic to the $(18, 18, 18; 36)$ triangle group from Corollary 1.3 from [15]. This triangle group is of type (p, p, p) , such triangle groups were discussed by Parker [23] and it is shown that an $(18, 18, 18; n)$ triangle group is discrete if and only if $n = 18$. Hence the $(18, 18, 36; 18)$ triangle group is non-discrete.

Sum (2.1.6): Suppose a solution to our equation contains (2.1.6) then five cosines sum to zero and one cosine is zero. That is, one of the angles is an odd multiple of $\pi/2$. As $p \geq 13$ and $n \geq 7$ it must be that one of x , y or z is a multiple of $\pi/2$ and the cosines $\cos \frac{\pi}{p}$, $\cos \frac{(p-4)\pi}{p}$ and $\cos \frac{2\pi}{n}$ belong to (2.1.6). Now if (2.1.6) involves $\cos \frac{\pi}{p}$ then we can assume that we have an equation of the form

$$0 = \cos \frac{\pi}{p} + \cos \left(\frac{\pi}{p} + \frac{2\pi}{5} \right) + \cos \left(\frac{\pi}{p} - \frac{2\pi}{5} \right) + \cos \left(\frac{\pi}{p} + \frac{4\pi}{5} \right) + \cos \left(\frac{\pi}{p} - \frac{4\pi}{5} \right),$$

But as $\frac{2\pi}{n}, \frac{\pi}{p} \in (0, \frac{\pi}{3})$ it is not possible for them to both belong to (2.1.6). Hence using (2.1.6) does not lead to a solution.

Sum (2.1.7): To check whether there was a solution to the equation using (2.1.7) I created a procedure in Maple. We first rewrite equation (2.1.7):

$$\begin{aligned} 0 = & \cos \left(\varphi + \frac{2\pi}{3} \right) + \cos \left(\varphi + \frac{4\pi}{3} \right) + \cos \left(\varphi + \frac{\pi}{5} \right) \\ & + \cos \left(\varphi + \frac{3\pi}{5} \right) + \cos \left(\varphi + \frac{7\pi}{5} \right) + \cos \left(\varphi + \frac{9\pi}{5} \right). \end{aligned}$$

Let $\mathcal{A} = \left[\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5} \right]$. If we use (2.1.7) for a solution then every angle in (2.2.1) can be written in the form $\varphi \pm \mathcal{A}[i]$ modulo 2π , where $\mathcal{A}[i]$ is an angle in \mathcal{A} . Our procedure runs through every possible combination and permutation. The only plausible result we obtain is that we have a solution when $\varphi = \frac{-8\pi}{15}$, which is when $p = 15$ and $n = 15$, which

leads to a $(15, 15, 30; 15)$ triangle group. But again using Corollary 1.3 from [15], the triangle group $(15, 15, 30; 15)$ is isomorphic to the triangle group $(15, 15, 15; 30)$, which is of type (p, p, p) . The (p, p, p) triangle groups were studied by Parker [23] and it is shown that any $(15, 15, 15)$ triangle group is non-discrete.

Since $p \geq 13$, then by inspection of Theorem 2.1.0.15, one of the following is true:

- a) $\cos \frac{\pi}{p}$ is involved in a vanishing sum of length at most two.
- b) $\cos \frac{\pi}{p}$ is involved in one of (2.1.5), (2.1.6) or (2.1.7).
- c) $\cos \frac{\pi}{p}$ is involved in one of (2.1.8), ..., (2.1.22).

We have already dealt with cases a) and b). In case c) we can see by looking down the list that either $p = 15$ or $p = 21$. In particular,

- If $p = 15$ then the sum must involve (2.1.11). In this case, there is no cosine that may take for $\cos \frac{2\pi}{n}$ as $n \geq 7$. Therefore we have the following two equations

$$\begin{aligned} \cos \frac{\pi}{3} - \cos \frac{\pi}{5} + \cos \frac{\pi}{15} - \cos \frac{4\pi}{15} &= 0 \\ \cos \frac{2\pi}{n} - \cos \frac{2\pi}{n} &= 0. \end{aligned}$$

In other words $x = \pm \frac{\pi}{3}$, $y = \pm \frac{4\pi}{5}$ and $z = \pm \left(\pi - \frac{2\pi}{n}\right)$. The only solution with $x + y + z = 2k\pi$ is $n = 15$. Hence for the $(15, 15, 30; 15)$ triangle group there exist $x, y, z \in \mathbb{Q}\pi$ such that we have a solution to equation (2.2.1) but this triangle group was discussed earlier and found as being non-discrete.

- If $p = 21$ then the sum must be (2.1.18), since of the sums involving $\frac{\pi}{21}$ this is the only one involving $\frac{4\pi}{21}$ as well. Note that (2.1.14) has length five, but since $\cos \frac{4\pi}{21} \neq 0$ we can exclude this case. By inspection from (2.1.18) there is no possibility for $\cos \frac{2\pi}{n}$ with $n \geq 7$. This rules this case out.

We have used all possible combinations of sums to try and solve equation (2.2.1) and the solutions we found were when $p = 15$ and $n = 20$ and when $n = 2p/3$. First we look at the $(15, 15, 30; 20)$ triangle group more closely.

Note that to obtain equation (2.2.1) we equated the trace formulae from [27] for w_A and w_B to a form that they should take if w_A and w_B are both regular elliptic and finite. We then rearranged the equations,

$$1 + 2 \cos \frac{2\pi}{n} = 16r_1^2 r_2^2 + 4r_3^2 - 1 - 16r_1 r_2 r_3 \cos \alpha \quad (2.1.1)$$

$$e^{ix} + e^{iy} + e^{iz} = 8r_1 r_2 r_3 e^{i\alpha} - 4(r_1^2 + r_2^2 + r_3^2) + 3, \quad (2.1.2)$$

in terms of $\cos \alpha$ and equated, but to do this we used the real part of (2.1.2). The next logical step would be to check whether the imaginary part of (2.1.2) is also satisfied for the given p, n and for all possible x, y and z that satisfy (2.2.1). We now find all such $x, y, z \in (-\pi, \pi)$.

To obtain a solution to (2.2.1) we used Theorem 2.1.0.15 (2.1.5) exactly once, when $\varphi = \frac{\pi}{15}$ (see above) and for $n = 20$ we have the following two equations

$$\begin{aligned} \cos \frac{\pi}{15} + \cos \frac{11\pi}{15} + \cos \frac{3\pi}{5} &= 0 \\ \text{and} \quad \cos \frac{\pi}{10} + \cos \frac{9\pi}{10} + \cos \frac{\pi}{2} &= 0, \end{aligned}$$

That is for $\{x, y, z\} \in \pm\{\frac{\pi}{2}, \frac{3\pi}{5}, \frac{9\pi}{10}\}$ we have a solution to equation (2.2.1) with $x_y + z = 2k\pi$, for $p = 15$ and $n = 20$. Fixing $p = 15$ and $n = 20$ determines the angular invariant α , which we find using (2.1.1) as

$$\alpha = 0.2630484959.$$

Now we need to check whether the imaginary part of (2.1.2) equates for the x, y

and z we found above. The imaginary part of the right hand side of (2.1.2) is

$$8r_1r_2r_3 \sin \alpha = 1.979378628.$$

If we check the imaginary part of the left hand side for the possible x, y and z ;

$$\sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{3\pi}{5}\right) + \sin\left(\frac{9\pi}{10}\right) = 2.26007351067,$$

we find that the imaginary part of (2.1.2) does not equate, therefore the $(15, 15, 30; 20)$ triangle group is not discrete. We can ignore the negative value as $8r_1r_2r_3 \sin \alpha > 0$.

We apply the same technique for all the triangle groups that have a solution to (2.2.1) when $n = 2p/3$. Recall that

$$\{x, y, z\} = \pm \left\{ \frac{4\pi}{p}, \left(\pi - \frac{\pi}{p} \right), \left(\pi - \frac{2\pi}{n} \right) \right\}.$$

For our given p and n we find the corresponding x, y, z and α and compute whether the imaginary part of (2.1.2) equates.

- $(15, 15, 30; 10)$. We have

$$\{x, y, z\} = \pm \left\{ \frac{4\pi}{15}, \frac{14\pi}{15}, \frac{4\pi}{5} \right\},$$

$$\alpha = 0.1776533050,$$

$$8r_1r_2r_3 \sin \alpha = 1.345239380,$$

$$\sin x + \sin y + \sin z = 1.538841769.$$

Therefore the $(15, 15, 30; 10)$ triangle group is not discrete.

- (18, 18, 36; 12). We have

$$\begin{aligned}\{x, y, z\} &= \pm \left\{ \frac{2\pi}{9}, \frac{17\pi}{18}, \frac{5\pi}{6} \right\}, \\ \alpha &= 0.1490077227, \\ 8r_1r_2r_3 \sin \alpha &= 1.47460089, \\ \sin x + \sin y + \sin z &= 1.316435787.\end{aligned}$$

Therefore the (18, 18, 36; 12) triangle group is not discrete.

- (21, 21, 42; 14). We have

$$\begin{aligned}\{x, y, z\} &= \pm \left\{ \frac{4\pi}{21}, \frac{20\pi}{21}, \frac{6\pi}{7} \right\}, \\ \alpha &= 0.1282134551, \\ 8r_1r_2r_3 \sin \alpha &= 0.9973807523, \\ \sin x + \sin y + \sin z &= 1.146246064.\end{aligned}$$

Therefore the (21, 21, 42; 14) triangle group is not discrete.

- (24, 24, 48; 16). We have

$$\begin{aligned}\{x, y, z\} &= \pm \left\{ \frac{1\pi}{6}, \frac{23\pi}{24}, \frac{7\pi}{8} \right\}, \\ \alpha &= 0.1124646558, \\ 8r_1r_2r_3 \sin \alpha &= 0.8806359740, \\ \sin x + \sin y + \sin z &= 1.013209625.\end{aligned}$$

Therefore the (24, 24, 48; 16) triangle group is not discrete.

- (27, 27, 54; 18). We have

$$\begin{aligned}\{x, y, z\} &= \pm \left\{ \frac{4\pi}{27}, \frac{26\pi}{27}, \frac{8\pi}{9} \right\}, \\ \alpha &= 0.1001371917, \\ 8r_1r_2r_3 \sin \alpha &= 0.7876457372, \\ \sin x + \sin y + \sin z &= 0.9069122376.\end{aligned}$$

Therefore the (27, 27, 54; 18) triangle group is not discrete.

- (30, 30, 60; 20). We have

$$\begin{aligned}\{x, y, z\} &= \pm \left\{ \frac{2\pi}{15}, \frac{29\pi}{30}, \frac{9\pi}{10} \right\}, \\ \alpha &= 0.09023168800, \\ 8r_1r_2r_3 \sin \alpha &= 0.7120208204, \\ \sin x + \sin y + \sin z &= 0.8202821006.\end{aligned}$$

Therefore the (30, 30, 60; 20) triangle group is not discrete.

[This proves Theorem 2.2.0.19]

□

We may translate Theorem 2.2.0.19 into the following table. Also included for comparison, are the areas of non-discreteness proved by using the Jørgensen inequalities. For the $(p, p, 2p; n)$ triangle group we have that $r_1 = r_2 = \cos \frac{\pi}{p}$ and $2r_3^2 = 2 \cos^2(\frac{\pi}{2p}) = \cos \frac{\pi}{p} + 1 = r_1 + 1$. The Jørgensen inequalities simplify to

$$\begin{aligned}\cos(2\pi/n) &< 8r_1^4 - 6r_1^2 - r_1 + \sqrt{1 - r_1^2}, \\ \cos(2\pi/n) &< 8r_1^4 - 8r_1^2 + 1 + \sqrt{1 - r_3^2}.\end{aligned}$$

This gives us the entries in the third column. For entries in the second column we

use the inequality that the triangle group $\Gamma(p, p, 2p; n)$ is degenerate if and only if

$$\cos \frac{2\pi}{n} < 8r_1^4 - 8r_1^2 - r_1 + 2.$$

Table 2.1:

Triangle Group	Degenerate	Non-discrete by Jørgensen Inequality	Non-discrete by Th. 2.2.0.19
$(13, 13, 26; n)$	$n \leq 6$	$n \leq 8$	$n = 7, n \geq 12$
$(14, 14, 28; n)$	$n \leq 7$	$n \leq 9$	$n \geq 13$
$(15, 15, 30; n)$	$n \leq 7$	$n \leq 10$	$n = 8, n \geq 14$
$(16, 16, 32; n)$	$n \leq 8$	$n \leq 11$	$n \geq 15$
$(17, 17, 34; n)$	$n \leq 8$	$n \leq 12$	$n = 9, n \geq 16$
$(18, 18, 36; n)$	$n \leq 9$	$n \leq 13$	$n = 10, n \geq 16$
$(19, 19, 38; n)$	$n \leq 9$	$n \leq 14$	$n = 10, n \geq 17$
$(20, 20, 40; n)$	$n \leq 10$	$n \leq 16$	$n = 11, n \geq 18$
$(21, 21, 42; n)$	$n \leq 10$	$n \leq 18$	$n = 11, n \geq 19$
$(22, 22, 44; n)$	$n \leq 11$	$n \leq 20$	$n = 12, n \geq 20$
$(23, 23, 46; n)$	$n \leq 11$	$n \leq 22$	$n = 12, n \geq 21$
$(24, 24, 48; n)$	$n \leq 12$	$n \leq 26$	$n = 13, n \geq 22$
$(25, 25, 50; n)$	$n \leq 12$	$n \leq 29$	$n = 13, n \geq 23$
$(26, 26, 52; n)$	$n \leq 13$	$n \leq 35$	$n = 14, n \geq 24$
$(27, 27, 54; n)$	$n \leq 13$	$n \leq 43$	$n = 14, 15, n \geq 25$
$(28, 28, 56; n)$	$n \leq 14$	$n \leq 58$	$n = 15, n \geq 25$
$(29, 29, 58; n)$	$n \leq 14$	$n \leq 104$	$n = 15, 16, n \geq 26$
$(30, 30, 60; n)$	$n \leq 15$	$n \geq 16$	$n = 16, n \geq 27$
$(31, 31, 62; n)$	$n \leq 16$	$n \geq 17$	$n = 17, n \geq 28$

Remark 2.2.0.21. In the table we go up to $p = 31$ as it is shown in [15] that for all $(p_1, p_2, p_3; n)$ triangle groups with $p_1 > 31$ the group is not discrete. In fact

from the table we can see that for $p \geq 30$ the Jørgensen inequality is stronger than the result of Theorem 2.2.0.19 and a $(p, p, 2p; n)$ triangle group with $p \geq 30$ is not discrete.

Remark 2.2.0.22. For $21 \leq p \leq 29$ we have that the Jørgensen inequality and the result of Theorem 2.2.0.19 fully complement each other so we may conclude that a $(p, p, 2p; n)$ triangle group with $p \geq 21$ is not discrete.

2.3 Triangle Groups with $\varphi_1 = \varphi_2 + \varphi_3$

In this section we study complex hyperbolic triangle groups that satisfy the equation

$$\frac{r_1^2 + r_2^2 + r_3^2 - 1}{2r_1 r_2 r_3} = 1 \iff r_1^2 + r_2^2 + r_3^2 = 1 + 2r_1 r_2 r_3. \quad (2.3.1)$$

This reduces the constraint on $\cos \alpha$ from Proposition 1.2.0.4 and the condition becomes $\cos \alpha < 1$ which is true for $\alpha \in (0, 2\pi)$. Assume $r_1 \leq r_2 \leq r_3$ and hence $\varphi_1 \geq \varphi_2 \geq \varphi_3 \geq 0$. Then equation (2.3.1) is equivalent to

$$\begin{aligned} & \cos^2 \varphi_1 + \cos^2 \varphi_2 + \cos^2 \varphi_3 = 1 + 2 \cos \varphi_1 \cos \varphi_2 \cos \varphi_3 \\ \iff & \cos^2 \varphi_1 + \frac{1}{2}(\cos 2\varphi_2 + \cos 2\varphi_3) + 1 = 1 + \cos \varphi_1 (\cos(\varphi_2 + \varphi_3) + \cos(\varphi_2 - \varphi_3)) \\ \iff & \cos^2 \varphi_1 + \cos(\varphi_2 + \varphi_3) \cos(\varphi_2 - \varphi_3) - \cos \varphi_1 (\cos(\varphi_2 + \varphi_3) + \cos(\varphi_2 - \varphi_3)) = 0 \\ \iff & (\cos \varphi_1 - \cos(\varphi_2 - \varphi_3)) (\cos \varphi_1 - \cos(\varphi_2 + \varphi_3)) = 0 \\ \iff & \cos \varphi_1 = \cos(\varphi_2 - \varphi_3) \quad \text{or} \quad \cos \varphi_1 = \cos(\varphi_2 + \varphi_3) \end{aligned}$$

Clearly if $\varphi_1 = \varphi_2 + \varphi_3$ then above is satisfied. Suppose

$$\cos \varphi_1 = \cos(\varphi_2 - \varphi_3),$$

then, because $\frac{\pi}{2} > \varphi_1 \geq \varphi_2 \geq \varphi_3 \geq 0$ we have $\varphi_1, (\varphi_2 - \varphi_3) \in [0, \frac{\pi}{2})$ and $\varphi_1, (\varphi_2 - \varphi_3)$ have the same cosine. Therefore it must be that $\varphi_1 = \varphi_2 - \varphi_3$, but $\varphi_1 \geq \varphi_2$ implies $\varphi_3 = 0$ and $\varphi_1 = \varphi_2$. Hence $\varphi_1 = \varphi_2 + \varphi_3$ is satisfied. Suppose

$$\cos \varphi_1 = \cos(\varphi_2 + \varphi_3),$$

then clearly $\varphi_1 = \varphi_2 + \varphi_3$ as $\frac{\pi}{2} > \varphi_1 \geq \varphi_2 \geq \varphi_3 \geq 0$. Therefore equation (2.3.1) is equivalent to $\varphi_1 = \varphi_2 + \varphi_3$ and we can simplify (2.1.3) to

$$1 = \cos(2\varphi_1 + 2\varphi_2) - \cos x - \cos y - \cos z - \cos \frac{2\pi}{n}. \quad (2.3.2)$$

where $x + y + z = 2k\pi, k \in \mathbb{Z}$. As stated in section 2.1 and [27, Section 11], the element w_A is regular elliptic if and only if $|t| > t_A$, where

$$t_A = \sqrt{\frac{1 + c_A}{1 - c_A}} \quad \text{and} \quad c_A = \frac{4r_1^2 r_2^2 + r_3^2 - 1}{4r_1 r_2 r_3}.$$

For $c_A > 1$ we set $t_A = \infty$. But for the triangle groups with $\varphi_1 = \varphi_2 + \varphi_3$ we have

$$\begin{aligned} c_A &= \frac{4 \cos^2 \varphi_1 \cos^2 \varphi_2 + \cos^2 \varphi_3 - 1}{4 \cos \varphi_1 \cos \varphi_2 \cos \varphi_3} \\ &= 1 + \frac{(2 \cos \varphi_1 \cos \varphi_2 - \cos \varphi_3)^2 - 1}{4 \cos \varphi_1 \cos \varphi_2 \cos \varphi_3} \\ &= 1 + \frac{(\cos(\varphi_1 + \varphi_2) + \cos(\varphi_1 - \varphi_2) - \cos \varphi_3)^2 - 1}{4 \cos \varphi_1 \cos \varphi_2 \cos \varphi_3} \\ &= 1 + \frac{\cos^2(\varphi_1 + \varphi_2) - 1}{4 \cos \varphi_1 \cos \varphi_2 \cos \varphi_3} \\ &= 1 - \frac{\sin^2(\varphi_1 + \varphi_2)}{4r_1 r_2 r_3} < 1. \end{aligned}$$

As mentioned earlier, to check the ellipticity of the element w_B is computationally much more intricate in general. In [27, Section 12], it is shown that the Goldman

function of the $\text{trace}(w_B)$ is of the form

$$f(\text{tr}(w_B)) = \frac{h(t)}{(t^2 + 1)^3},$$

where h is an even polynomial of degree at most six and $t \in [0, \infty)$. Hence w_B is elliptic at t_0 if and only if $h(t_0) < 0$. For the triangle groups satisfying $\varphi_1 = \varphi_2 + \varphi_3$ the explicit computation of h is simpler, [27, section 12]. We obtain

$$\frac{h(t)}{1024R} = (1 - R)t^4 + (64R^3 - 80R^2 + 11R + 2)t^2 + (64R^3 + 48R^2 + 12R + 1),$$

where $R = r_1 r_2 r_3$. The polynomial h has no roots if $R < 7/8$. For $R \geq 7/8$ the roots of the polynomial h can be computed as t_{\pm} , where

$$t_{\pm} = \sqrt{\frac{64R^3 - 80R^2 + 11R + 2 \pm R\sqrt{(8R - 7)^3(8R + 1)}}{2(R - 1)}}.$$

That is

- for $R < 7/8$ we have $h(t) > 0$ for all $t \in [0, \infty)$,
- for $R = 7/8$ we have $t_+ = t_-$ and $h(t) > 0$ for all $t \in [0, t_+) \cup (t_+, \infty)$,
- for $7/8 < R < 1$ we have $h(t) > 0$ for $t < t_+$ or $t > t_-$ and $h(t) < 0$ for $t_+ < t < t_-$,
- for $R > 1$ we have $h(t) > 0$ for $t < t_-$ and $h(t) < 0$ for $t > t_-$.

In the case of the ideal triangle group we have $R = 1$ and hence (compare [10])

$$f(\text{tr}(w_B)) = \frac{1024(125 - 3t^2)}{(t^2 + 1)^3}.$$

For $R \leq 1$ we have $r_k = \cos \varphi_k$ and for w_B to be elliptic we require $r_1 r_2 r_3 \geq 7/8$.
Now

$$\begin{aligned}
 \cos \varphi_1 \cos \varphi_2 \cos \varphi_3 &\geq \frac{7}{8} \\
 \iff \cos \varphi_1 (\cos (\varphi_2 + \varphi_3) + \cos (\varphi_2 - \varphi_3)) &\geq \frac{7}{4} \\
 \iff \cos (\varphi_2 - \varphi_3) &\geq \frac{7}{4 \cos \varphi_1} - \cos \varphi_1, \\
 \text{But } \frac{7}{4 \cos \varphi_1} - \cos \varphi_1 &> 1, \quad \text{for } 2 < p \leq 7.
 \end{aligned}$$

Hence for $p \leq 7$, the element w_B is never elliptic.

According to Proposition 2.1.0.9, the triangle group is non-degenerate if and only if $\text{Re}(\text{tr}(w_B)) < -1$, that is

$$\begin{aligned}
 \cos x + \cos y + \cos z &= \cos(2\varphi_1 + 2\varphi_2) - \cos \frac{2\pi}{n} - 1 < -1 \\
 \iff \cos(2\varphi_1 + 2\varphi_2) &< \cos \frac{2\pi}{n} \\
 \iff \pi - \pi/n > \varphi_1 + \varphi_2 > \pi/n &\quad \text{as } \varphi_k, \frac{2\pi}{n} \in (0, \pi/2).
 \end{aligned}$$

We can rewrite equation (2.3.2) as

$$0 = -2 \cos \frac{\pi}{3} + \cos(2\varphi_1 + 2\varphi_2) - \cos x - \cos y - \cos z - \cos \frac{2\pi}{n},$$

which is a rational linear combination of six cosines of rational multiples of π that vanish. Therefore, using the results of Parker we see that there is no rational linear combination of at most six cosines of rational multiples of π strictly between 0 and $\pi/2$, whose sum vanishes but no proper part of the linear combination vanishes, of this form. Hence equation (2.3.2) must split into shorter linear combinations of cosines which are rational. One such linear combination has a maximum of four cosines. Therefore we can use Theorem 7 in Conway and Jones [2] to find combinations of such rational sums of cosines, with special care that our angles

do not necessarily lie between 0 and $\pi/2$. This gives us a list of sums of cosines we can use together with known rational cosines to solve equation (2.3.2). Rather than write the theorem, which is similar to Theorem 2.1.0.15, we just state the sums needed for this case:

$$0 = \sum_{j=0}^2 \cos\left(\varphi + \frac{2j\pi}{3}\right) \quad (\text{i})$$

$$\frac{1}{2} = \cos \frac{\pi}{3} \quad (\text{ii})$$

$$\frac{1}{2} = \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} \quad (\text{iii})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} \quad (\text{iv})$$

$$\frac{1}{2} = \cos \frac{\pi}{5} - \cos \frac{\pi}{15} + \cos \frac{4\pi}{15} \quad (\text{v})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{5} + \cos \frac{2\pi}{15} - \cos \frac{7\pi}{15} \quad (\text{vi})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{2\pi}{21} - \cos \frac{5\pi}{21} \quad (\text{vii})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} + \cos \frac{3\pi}{7} - \cos \frac{\pi}{21} + \cos \frac{8\pi}{21} \quad (\text{viii})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{4\pi}{21} + \cos \frac{10\pi}{21} \quad (\text{ix})$$

$$\frac{1}{2} = -\cos \frac{\pi}{15} + \cos \frac{2\pi}{15} + \cos \frac{4\pi}{15} - \cos \frac{7\pi}{15} \quad (\text{x})$$

Let $\mathbf{A} := \{(\text{i}), \dots, (\text{x})\}$. Note that if $p \geq 10$ and finite then $2\varphi_1 + 2\varphi_2 \leq \frac{2\pi}{5}$. Therefore $\cos(2\varphi_1 + 2\varphi_2)$ is not rational, and hence $\cos(\frac{2\pi}{n})$ is also not rational. We need to find solutions to equation (2.3.2) using the set of equations \mathbf{A} . We bear in mind that our equation can have subsets which are rational, the angles are not necessarily distinct and do not necessarily lie in $(0, \frac{\pi}{2})$. If we have coinciding angles then we may rewrite equation (2.3.2) in the form

$$1 = A_1 \cos a_1\pi + A_2 \cos a_2\pi + A_3 \cos a_3\pi + \dots$$

where $A_i \in \mathbb{Z}$, $a_i \in \mathbb{Q}$ with some $A_i > 1$. From the list of sums in the set \mathbf{A} we have no minimal sum of this form so we must therefore combine two smaller sums. Consider the case when one of the angles is $2k\pi$ for some $k \in \mathbb{Z}$, then without loss

of generality we may assume $x = 2k\pi$. This then implies that

$$\mathrm{tr}(w_B) = 1 + 2 \cos y \in (-1, 3),$$

which is a contradiction as $\mathrm{tr}(w_B) < -1$. Similarly if we assume $x = \pi + 2k\pi$ then $\mathrm{tr}(w_B) = -1$, which is not possible. Therefore when looking for solutions to equation (2.3.2) we cannot use the angle $k\pi$.

Proposition 2.3.0.23. *The complex hyperbolic triangle group $\Gamma(p, 2p, 2p; n)$ is non-discrete for all $p \in \mathbb{Z}$ when $\Gamma(p, 2p, 2p; n)$ is of type B and w_B is regular elliptic.*

Remark 2.3.0.24. *The fact we consider the triangle group $\Gamma(p, 2p, 2p; n)$ with n finite implies that w_A is elliptic.*

Remark 2.3.0.25. *Using the trace formulas and the discriminant function f described in section 1.1.2, we can calculate that the $(p, 2p, 2p)$ triangle group is of type B if $p \geq 10$.*

Proof. In the $(p, 2p, 2p)$ case, equation (2.3.2) becomes

$$1 = \cos \frac{3\pi}{p} - \cos x - \cos y - \cos z - \cos \frac{2\pi}{n}, \quad (2.3.3)$$

with $\frac{3\pi}{p} > \frac{2\pi}{n}$ and $x + y + z = 2k\pi, k \in \mathbb{Z}$. As $\varphi_1 + \varphi_2 > \frac{\pi}{n}$ from the condition above and $p \geq 10$ then $\frac{2\pi}{n} < \frac{3\pi}{p} \leq \frac{3\pi}{10} \Rightarrow n \geq 7$. We need to find solutions to this equation using the set **A**.

Suppose (2.3.3) contains a sum which vanishes. Clearly this vanishing sum must be less than four as we have no angle of the form $k\pi$. If one cosine vanishes then we are left with four cosines which sum to one, this will be covered when we look at rational sums. Assume we have at most a pair of cosines that vanish. Note that

$$\cos \frac{3\pi}{p} < \cos \frac{2\pi}{n},$$

hence this pair cannot vanish together and clearly a pair from $\cos x$, $\cos y$ and $\cos z$ cannot vanish as $\text{tr}(w_B) < -1$. So either (2.3.3) involves the sum

$$\begin{aligned} \cos \frac{3\pi}{p} - \cos \frac{3\pi}{p} &= 0 \\ \text{or} \quad \cos \frac{2\pi}{n} + \cos(\pi - \frac{2\pi}{n}) &= 0. \end{aligned}$$

The remaining three cosines are of the form

$$\pm \cos a - \cos b - \cos c = 1,$$

where $a, b, c \in \mathbb{Q}\pi$. These three cosines cannot each be rational themselves as one of them is either $\cos(\frac{3\pi}{p})$ or $-\cos \frac{2\pi}{n}$. Therefore it must be that we combine (ii) and (iii) for a solution.

$$\frac{1}{2} = \cos \frac{\pi}{3} \tag{ii}$$

$$\frac{1}{2} = \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} \tag{iii}$$

Notice that there is no cosine of the form $-\cos \frac{2\pi}{n}$, $n \geq 7$, and the only cosine of the form $\cos(\frac{\pi}{p})$, $p \geq 10$, is $\cos \frac{\pi}{5}$. Therefore we must use the following three sums for a solution

$$0 = \cos \frac{2\pi}{n} + \cos(\pi - \frac{2\pi}{n})$$

$$\frac{1}{2} = \cos \frac{\pi}{5} - \cos \frac{2\pi}{5}$$

$$\frac{1}{2} = \cos \frac{\pi}{3}.$$

To check whether the remaining angles can be of the form x , y and z we search for solutions to

$$\left(\pi - \frac{2\pi}{n}\right) \pm \frac{2\pi}{5} \pm \frac{2\pi}{3} = 2k\pi,$$

where $k \in \mathbb{Z}$. This has a solution in the $(++)$ case for $n = 30$. Therefore the $(15, 30, 30; 30)$ triangle group may be discrete. We return to this later in the proof.

Suppose a solution contains (i), that is three of our cosines sum to zero and the remaining two sum to 1. Then we are left with an equation of the form

$$\pm \cos a - \cos b = 1,$$

where $a, b \in \pi\mathbb{Q}$. Solutions to this equation are only possible if a and b are multiples of $\pi/3$. But $\frac{3\pi}{p}, \frac{2\pi}{n} \in (0, \frac{\pi}{3})$, so a and b are not of the form $\frac{3\pi}{p}$ or $\frac{2\pi}{n}$ and therefore $\cos \frac{3\pi}{p}$ and $\cos \frac{2\pi}{n}$ must be used in (i). Without a loss of generality we may assume that $\cos x$ is involved in (i). That is

$$\begin{aligned} 0 &= \cos\left(\pi - \frac{3\pi}{p}\right) + \cos \frac{2\pi}{n} + \cos x \\ 1 &= -\cos y - \cos z \end{aligned}$$

Therefore $x = \pi \pm \frac{3\pi}{p} \pm \frac{2\pi}{3}$, $y = \pm \frac{2\pi}{3}$ and $z = \pm \frac{2\pi}{3}$. We need to check $x + y + z = 2k\pi$ for some $k \in \mathbb{Z}$. That is equivalent to checking

$$\frac{3\pi}{p} \pm \frac{2\pi}{3} \pm \frac{2\pi}{3} \pm \frac{2\pi}{3} = (2k - 1)\pi$$

which is not possible as $\frac{3\pi}{p} \in (0, \frac{\pi}{3})$.

So it must be that we combine equations from (ii)-(x). First we mark out the cosines on our list that can be of the form $\cos \frac{3\pi}{p}$, $p \geq 10$, in red and those of the

form $-\cos \frac{2\pi}{n}$, $n \geq 7$, in blue,

$$\frac{1}{2} = \cos \frac{\pi}{3} \quad (\text{ii})$$

$$\frac{1}{2} = \text{red} \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} \quad (\text{iii})$$

$$\frac{1}{2} = \text{red} \cos \frac{\pi}{7} - \text{blue} \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} \quad (\text{iv})$$

$$\frac{1}{2} = \text{red} \cos \frac{\pi}{5} - \text{blue} \cos \frac{\pi}{15} + \cos \frac{4\pi}{15} \quad (\text{v})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{5} + \cos \frac{2\pi}{15} - \cos \frac{7\pi}{15} \quad (\text{vi})$$

$$\frac{1}{2} = \text{red} \cos \frac{\pi}{7} - \text{blue} \cos \frac{2\pi}{7} + \cos \frac{2\pi}{21} - \cos \frac{5\pi}{21} \quad (\text{vii})$$

$$\frac{1}{2} = \text{red} \cos \frac{\pi}{7} + \cos \frac{3\pi}{7} - \text{blue} \cos \frac{\pi}{21} + \cos \frac{8\pi}{21} \quad (\text{viii})$$

$$\frac{1}{2} = -\text{blue} \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{4\pi}{21} + \cos \frac{10\pi}{21} \quad (\text{ix})$$

$$\frac{1}{2} = -\text{blue} \cos \frac{\pi}{15} + \cos \frac{2\pi}{15} + \cos \frac{4\pi}{15} - \cos \frac{7\pi}{15} \quad (\text{x})$$

Notice that we need to include a rational sum of length at least three so that we have a cosine of the form $-\cos \frac{2\pi}{n}$, $n \geq 7$.

Sums of Length 3: If a solution contains a sum of length three then we are left with the remaining two cosines

$$\pm \cos a - \cos b = \frac{1}{2}. \quad (2.3.4)$$

Solutions to this are given when either one cosine is $\pm \frac{1}{2}$ and the other is 0 or we use sum (iii). Therefore we combine sums of length three with either of these possibilities to find suitable solutions for p , n , x and y .

First consider the sum (iv), it must be that $\frac{2\pi}{n} = \frac{2\pi}{7}$ which implies that we cannot have $\frac{3\pi}{p} = \frac{\pi}{7}$ as $\frac{3\pi}{p} > \frac{2\pi}{n}$. Therefore we must combine (iv) with (iii) and $\frac{3\pi}{p} = \frac{\pi}{5}$. This leaves us with the remaining three cosines

$$\cos \frac{\pi}{7} + \cos \frac{3\pi}{7} - \cos \frac{2\pi}{5} = -\cos x - \cos y - \cos z.$$

Therefore to check whether the angles can be written in the form x , y and z we search for solutions to

$$\frac{6\pi}{7} \pm \frac{4\pi}{7} \pm \frac{2\pi}{5} = 2k\pi,$$

where $k \in \mathbb{Z}$. Taking the four possible sign combinations, it is simple enough to check that this has no solutions.

Next consider the sum (v), if this is used for a solution then $\frac{2\pi}{n} = \frac{\pi}{15}$ and we must combine this sum with possible solutions to (2.3.4). In all cases there is only one possibility for $\frac{3\pi}{p}$, that is $\frac{3\pi}{p} = \frac{\pi}{5}$. For the remaining three cosines, one must be equal to $\cos \frac{4\pi}{15}$ and the other two must satisfy (2.3.4). That is we have two possibilities

$$\cos \frac{4\pi}{15} + \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} = -\cos x - \cos y - \cos z$$

$$\cos \frac{4\pi}{15} - \cos \frac{2\pi}{3} - \cos \frac{\pi}{2} = -\cos x - \cos y - \cos z$$

We must now check whether $x + y + z = 2k\pi$, for $k \in \mathbb{Z}$. So we search for solutions to

$$\frac{11\pi}{15} \pm \frac{4\pi}{5} \pm \frac{2\pi}{5} = 2k\pi$$

$$\text{and} \quad \frac{11\pi}{15} \pm \frac{2\pi}{3} \pm \frac{\pi}{2} = 2k\pi.$$

It is easily checked that there are no such solutions.

Sum (vi) cannot be used as there is no cosine of the form $-\cos \frac{2\pi}{n}$, $n \geq 7$.

Sums of Length 4: If we use a sum of length four then it must be combined with (ii) so that equation (2.3.3) is satisfied. We cannot use (vii) as a solution as it does not contain cosines of the form $\cos \frac{3\pi}{p}$ or $-\cos \frac{2\pi}{n}$ that satisfy $\frac{3\pi}{p} > \frac{2\pi}{n}$. We also cannot use (ix) and (x) as a solution as neither sum contains any cosines of the form $\cos \frac{3\pi}{p}$.

Consider using (viii) as a solution then $\frac{3\pi}{p} = \frac{\pi}{7}$ and $\frac{2\pi}{n} = \frac{\pi}{21}$. The remaining three cosines are then

$$\cos \frac{\pi}{3} + \cos \frac{3\pi}{7} + \cos \frac{8\pi}{21} = -\cos x - \cos y - \cos z$$

Therefore to check whether the angles can be written in the form x , y and z so that $x + y + z = 2k\pi$, where $k \in \mathbb{Z}$, we search for solutions to

$$\frac{2\pi}{3} \pm \frac{4\pi}{7} \pm \frac{13\pi}{21} = 2k\pi.$$

Taking the four possible sign combinations we can easily check that there are no solutions.

We have therefore checked all possible solutions to equation (2.3.3) and found only one suitable answer; when $p = 15$, $n = 30$ and $\{x, y, z\} = \pm\{\frac{14\pi}{15}, \frac{2\pi}{5}, \frac{2\pi}{3}\}$. Fixing $p = 15$ and $n = 30$ determines the angular invariant α , which we find using (2.1.1) as

$$\alpha = 0.2094395099.$$

Now we need to check whether the imaginary part of (2.1.2) equates for the x , y and z we found above. The imaginary part of the right hand side of (2.1.2) is

$$8r_1r_2r_3 \sin \alpha = 1.609170235.$$

If we check the imaginary part of the left hand side for the possible x , y and z ;

$$\sin\left(\frac{14\pi}{15}\right) + \sin\left(\frac{2\pi}{5}\right) + \sin\left(\frac{2\pi}{3}\right) = 2.0249993611,$$

we find that the imaginary part of (2.1.2) does not equate, therefore the $(15, 30, 30; 30)$ triangle group is not discrete. We can ignore the negative value as

$$8r_1r_2r_3 \sin \alpha > 0.$$

This proves the proposition. \square

We can rewrite Proposition 2.3.0.23 in the following table. Also included for comparison, are the areas of non-discreteness proved by using the Jørgensen inequalities. For the $(p, 2p, 2p; n)$ triangle group we have that $2r_2^2 = 2r_3^2 = 2\cos^2(\frac{\pi}{2p}) = \cos \frac{\pi}{p} + 1 = r_1 + 1$. The Jørgensen inequalities then simplify to

$$\begin{aligned} \cos(2\pi/n) &< 4r_1^3 + 2r_1^2 - 3r_1 - 2 + \sqrt{1 - r_1^2}, \\ \cos(2\pi/n) &< 4r_1^3 - 2r_1 - 1 + \sqrt{1 - r_2^2}. \end{aligned}$$

This gives us the entries in the third column. For entries in the second column we use the inequality that the triangle group $\Gamma(p, 2p, 2p; n)$ is degenerate if and only if

$$\cos \frac{2\pi}{n} < r_1(4r_1^2 - 3).$$

Table 2.2:

Triangle Group	Degenerate	Non-discrete by Jørgensen Inequality	Non-discrete by Prop. 2.3.0.23
$(10, 20, 20; n)$	$n \leq 6$	$n \leq 7$	$n \geq 16$
$(11, 22, 22; n)$	$n \leq 7$	$n \leq 9$	$n \geq 18$
$(12, 24, 24; n)$	$n \leq 8$	$n \leq 10$	$n \geq 21$
$(13, 26, 26; n)$	$n \leq 8$	$n \leq 12$	$n \geq 23$
$(14, 28, 28; n)$	$n \leq 9$	$n \leq 14$	$n \geq 25$
$(15, 30, 30; n)$	$n \leq 10$	$n \leq 16$	$n \geq 27$
$(16, 32, 32; n)$	$n \leq 10$	$n \leq 19$	$n \geq 29$
$(17, 34, 34; n)$	$n \leq 11$	$n \leq 24$	$n \geq 31$
$(18, 36, 36; n)$	$n \leq 12$	$n \leq 30$	$n \geq 34$
$(19, 38, 38; n)$	$n \leq 12$	$n \leq 44$	$n \geq 36$
$(20, 40, 40; n)$	$n \leq 13$	$n \leq 114$	$n \geq 38$
$(21, 42, 42; n)$	$n \leq 14$	$n \geq 15$	$n \geq 40$
$(22, 44, 44; n)$	$n \leq 14$	$n \geq 15$	$n \geq 42$
$(23, 46, 46; n)$	$n \leq 15$	$n \geq 16$	$n \geq 44$
$(24, 48, 48; n)$	$n \leq 16$	$n \geq 17$	$n \geq 46$
$(25, 50, 50; n)$	$n \leq 16$	$n \geq 17$	$n \geq 48$
$(26, 52, 52; n)$	$n \leq 17$	$n \geq 18$	$n \geq 50$
$(27, 54, 54; n)$	$n \leq 18$	$n \geq 19$	$n \geq 52$
$(28, 56, 56; n)$	$n \leq 18$	$n \geq 19$	$n \geq 54$
$(29, 58, 58; n)$	$n \leq 19$	$n \geq 20$	$n \geq 57$
$(30, 60, 60; n)$	$n \leq 20$	$n \geq 21$	$n \geq 59$
$(31, 62, 62; n)$	$n \leq 20$	$n \geq 21$	$n \geq 61$

Remark 2.3.0.26. *In the table we go up to $p = 31$ as it is shown in [15] that for all $(p_1, p_2, p_3; n)$ triangle groups with $p_1 > 31$ the group is not discrete. In fact from the table we can see that for $p \geq 21$ the Jørgensen inequality is stronger than the result of Proposition 2.3.0.23 and a $(p, p, 2p; n)$ triangle group with $p \geq 21$ is not discrete.*

Remark 2.3.0.27. *For $p = 19$ and $p = 20$ we have that the Jørgensen inequality and the result of Proposition 2.3.0.23 fully complement each other so we may conclude that a $(p, 2p, 2p; n)$ triangle group with $p \geq 19$ is not discrete.*

Proposition 2.3.0.28. *The complex hyperbolic triangle group $\Gamma(p, 3p/2, 3p; n)$ is non-discrete for all $p \in \mathbb{Z}$ when $\Gamma(p, 3p/2, 3p; n)$ is of type B and w_B is regular elliptic.*

Remark 2.3.0.29. *The fact we consider the triangle group $\Gamma(p, 3p/2, 3p; n)$ with n finite implies that w_A is elliptic.*

Remark 2.3.0.30. *Using the trace formulas and the discriminant function f described in section 1.1.2, we can calculate that the $(p, 3p/2, 3p)$ triangle group is of type B if and only if $p \geq 11$.*

Proof. In the $(p, 3p/2, 3p)$ case, equation (2.3.2) becomes

$$1 = \cos \frac{10\pi}{3p} - \cos x - \cos y - \cos z - \cos \frac{2\pi}{n}, \quad (2.3.5)$$

with $\frac{10\pi}{3p} > \frac{2\pi}{n}$ and $x + y + z = 2k\pi, k \in \mathbb{Z}$. As $p \geq 11$ then $\frac{2\pi}{n} < \frac{10\pi}{3p} \leq \frac{10\pi}{33} \Rightarrow n \geq 7$. Again we look for solutions to this equation using the set **A**. Taking care again that our equation can have subsets which are rational, the angles are not necessarily distinct and do not necessarily lie in $(0, \frac{\pi}{2})$.

Suppose (2.3.5) contains a sum which vanishes. Clearly this vanishing sum must be less than four as we have no angle of the form $k\pi$. If one cosine vanishes then we are left with four cosines which sum to one, this will be covered when

we look at rational sums. Assume we have at most a pair of cosines that vanish. Note that

$$\cos \frac{10\pi}{3p} < \cos \frac{2\pi}{n},$$

hence this pair cannot vanish together and clearly a pair from $\cos x$, $\cos y$ and $\cos z$ cannot vanish as $\text{tr}(w_B) < -1$. So either (2.3.5) involves the sum

$$\begin{aligned} & \cos \frac{10\pi}{3p} - \cos \frac{10\pi}{3p} = 0 \\ \text{or} \quad & \cos \frac{2\pi}{n} + \cos(\pi - \frac{2\pi}{n}) = 0. \end{aligned}$$

The remaining three cosines are of the form

$$\pm \cos a - \cos b - \cos c = 1,$$

where $a, b, c \in \mathbb{Q}\pi$. These three cosines cannot each be rational themselves as one of them is either $\cos \frac{3\pi}{p}$ or $-\cos \frac{2\pi}{n}$. Therefore it must be that we combine (ii) and (iii) for a solution.

$$\frac{1}{2} = \cos \frac{\pi}{3} \tag{ii}$$

$$\frac{1}{2} = \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} \tag{iii}$$

Notice that there is no cosine of the form $-\cos \frac{2\pi}{n}$, $n \geq 7$, or $\cos \frac{10\pi}{3p}$, $p \geq 10$. Therefore there is no solution to (2.3.5) if a pair of cosines vanish.

For the same reasons discussed in the proof of Proposition 2.3.0.23 we cannot use (i) for a solution as $\frac{10\pi}{3p} \in (0, \frac{\pi}{3})$.

Therefore we procede as in the proof of Proposition 2.3.0.23 and we mark out the cosines on our list that can be of the form $\cos \frac{10\pi}{3p}$, $p \geq 11$, in red and those of the form $-\cos \frac{2\pi}{n}$, $n \geq 7$, in blue,

$$\frac{1}{2} = \cos \frac{\pi}{3} \quad (\text{ii})$$

$$\frac{1}{2} = \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} \quad (\text{iii})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} \quad (\text{iv})$$

$$\frac{1}{2} = \cos \frac{\pi}{5} - \cos \frac{\pi}{15} + \cos \frac{4\pi}{15} \quad (\text{v})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{5} + \cos \frac{2\pi}{15} - \cos \frac{7\pi}{15} \quad (\text{vi})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{2\pi}{21} - \cos \frac{5\pi}{21} \quad (\text{vii})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} + \cos \frac{3\pi}{7} - \cos \frac{\pi}{21} + \cos \frac{8\pi}{21} \quad (\text{viii})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{4\pi}{21} + \cos \frac{10\pi}{21} \quad (\text{ix})$$

$$\frac{1}{2} = -\cos \frac{\pi}{15} + \cos \frac{2\pi}{15} + \cos \frac{4\pi}{15} - \cos \frac{7\pi}{15} \quad (\text{x})$$

Notice that to obtain a solution to equation (2.3.5) we need to use (x) so that we have an cosine of the form $\cos \frac{10\pi}{3}$ and $-\cos \frac{2\pi}{n}$ with $\frac{10\pi}{3} > \frac{2\pi}{n}$. We then need to combine this with (ii), which leaves us with the following three cosines that need to be of the form

$$\cos \frac{\pi}{3} + \cos \frac{4\pi}{15} - \cos \frac{7\pi}{15} = -\cos x - \cos y - \cos z.$$

where $x + y + z = 2k\pi$, $k \in \mathbb{Z}$. That is we check for solutions to

$$\frac{2\pi}{3} \pm \frac{11\pi}{15} \pm \frac{7\pi}{15} = 2k\pi.$$

Taking the four possible choices in sign we find no solutions. Therefore the complex hyperbolic triangle group $\Gamma(p, 3p/2, 3p; n)$ is non-discrete for all $p \in \mathbb{Z}$ when $\Gamma(p, 3p/2, 3p; n)$ is of type B and w_B is regular elliptic.

□

Just as we did for Proposition 2.3.0.23 we can rewrite Proposition 2.3.0.28 in the following table. Also included for comparison, are the areas of non-discreteness proved by using the Jørgensen inequalities.

Table 2.3:

Triangle Group	Degenerate	Non-discrete by Jørgensen Inequality	Non-discrete by Prop. 2.3.0.28
$(11, \frac{33}{2}, 33; n)$	$n \leq 6$	$n \leq 7$	$n \geq 13$
$(12, 18, 36; n)$	$n \leq 7$	$n \leq 9$	$n \geq 14$
$(13, \frac{39}{2}, 39; n)$	$n \leq 7$	$n \leq 10$	$n \geq 15$
$(14, 21, 42; n)$	$n \leq 8$	$n \leq 11$	$n \geq 17$
$(15, \frac{45}{2}, 45; n)$	$n \leq 9$	$n \leq 13$	$n \geq 18$
$(16, 24, 48; n)$	$n \leq 9$	$n \leq 14$	$n \geq 19$
$(17, \frac{51}{2}, 51; n)$	$n \leq 10$	$n \leq 16$	$n \geq 20$
$(18, 27, 54; n)$	$n \leq 10$	$n \leq 19$	$n \geq 21$
$(19, \frac{57}{2}, 57; n)$	$n \leq 11$	$n \leq 22$	$n \geq 23$
$(20, 30, 60; n)$	$n \leq 12$	$n \leq 27$	$n \geq 24$
$(21, \frac{63}{2}, 63; n)$	$n \leq 12$	$n \leq 33$	$n \geq 25$
$(22, 33, 66; n)$	$n \leq 13$	$n \leq 45$	$n \geq 26$
$(23, \frac{69}{2}, 69; n)$	$n \leq 13$	$n \leq 84$	$n \geq 28$
$(24, 36, 72; n)$	$n \leq 14$	$n \geq 15$	$n \geq 29$
$(25, \frac{75}{2}, 75; n)$	$n \leq 15$	$n \geq 16$	$n \geq 30$
$(26, 39, 78; n)$	$n \leq 15$	$n \geq 16$	$n \geq 31$
$(27, \frac{81}{2}, 81; n)$	$n \leq 16$	$n \geq 17$	$n \geq 32$
$(28, 42, 84; n)$	$n \leq 16$	$n \geq 17$	$n \geq 34$
$(29, \frac{87}{2}, 87; n)$	$n \leq 17$	$n \geq 18$	$n \geq 35$
$(30, 45, 90; n)$	$n \leq 18$	$n \geq 19$	$n \geq 36$
$(31, \frac{93}{2}, 93; n)$	$n \leq 18$	$n \geq 19$	$n \geq 37$

Remark 2.3.0.31. *In the table we go up to $p = 31$ as it is shown in [15] that for all $(p_1, p_2, p_3; n)$ triangle groups with $p_1 > 31$ the group is not discrete. In fact from the table we can see that for $p \geq 24$ the Jørgensen inequality is stronger than the result of Proposition 2.3.0.28 and a $(p, 3p/2, 3p; n)$ triangle group with $p \geq 24$ is not discrete.*

Remark 2.3.0.32. *For $19 \leq p \leq 23$ we have that the Jørgensen inequality and the result of Proposition 2.3.0.28 fully complement each other so we may conclude that a $(p, 3p/2, 3p; n)$ triangle group with $p \geq 19$ is not discrete.*

Proposition 2.3.0.33. *The complex hyperbolic triangle group $\Gamma(p, 2p, 2p; n)$ is non-discrete for all $p \in \mathbb{Q}$, $p \geq 10$ and w_B is regular elliptic.*

Remark 2.3.0.34. *The fact we consider the triangle group $\Gamma(p, 2p, 2p; n)$ with n finite implies that w_A is elliptic.*

Proof. Like in Proposition 2.3.0.23 we see that equation (2.3.2) becomes

$$1 = \cos \frac{3\pi}{p} - \cos x - \cos y - \cos z - \cos \frac{2\pi}{n}, \quad (2.3.3)$$

for the triangle group $(p, 2p, 2p)$. We also have the same conditions $p \geq 10$, $7 \leq n \in \mathbb{Z}$ and $\frac{2\pi}{n} < \frac{3\pi}{p}$. We need to find solutions to this equation using the set **A**. Taking care again that our equation can have subsets which are rational, the angles are not necessarily distinct and do not necessarily lie in $(0, \frac{\pi}{2})$.

Suppose (2.3.3) contains a sum which vanishes. Clearly this vanishing sum must be less than four as we have no angle of the form $k\pi$. If one cosine vanishes then we are left with four cosines which sum to one, this will be covered when we look at rational sums. If we assume we have at most a pair of cosines that vanish then as $\frac{3\pi}{p} \in (0, \frac{\pi}{3})$ we will lead to the same solution we obtained in Proposition 2.3.0.23, where we have a solution for $\frac{3\pi}{p} = \frac{\pi}{5}$ and $\frac{2\pi}{n} = \frac{\pi}{15}$ but then in Proposition 2.3.0.23 we investigate further to find the $(15, 30, 30; 30)$ triangle group is not discrete.

For the same reasons discussed in the proof of Proposition 2.3.0.23 we cannot use (i) for a solution.

Therefore we procede as in the proof of Proposition 2.3.0.23 and we mark out the cosines on our list that can be of the form $\cos \frac{3\pi}{p}$, $p \geq 10$, in red and those of the form $-\cos \frac{2\pi}{n}$, $n \geq 7$, in blue,

$$\frac{1}{2} = \cos \frac{\pi}{3} \quad (\text{ii})$$

$$\frac{1}{2} = \textcolor{red}{\cos \frac{\pi}{5}} - \cos \frac{2\pi}{5} \quad (\text{iii})$$

$$\frac{1}{2} = \textcolor{red}{\cos \frac{\pi}{7}} - \textcolor{blue}{\cos \frac{2\pi}{7}} + \cos \frac{3\pi}{7} \quad (\text{iv})$$

$$\frac{1}{2} = \textcolor{red}{\cos \frac{\pi}{5}} - \textcolor{blue}{\cos \frac{\pi}{15}} + \textcolor{red}{\cos \frac{4\pi}{15}} \quad (\text{v})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{5} + \textcolor{red}{\cos \frac{2\pi}{15}} - \cos \frac{7\pi}{15} \quad (\text{vi})$$

$$\frac{1}{2} = \textcolor{red}{\cos \frac{\pi}{7}} - \textcolor{blue}{\cos \frac{2\pi}{7}} + \textcolor{red}{\cos \frac{2\pi}{21}} - \cos \frac{5\pi}{21} \quad (\text{vii})$$

$$\frac{1}{2} = \textcolor{red}{\cos \frac{\pi}{7}} + \cos \frac{3\pi}{7} - \textcolor{blue}{\cos \frac{\pi}{21}} + \cos \frac{8\pi}{21} \quad (\text{viii})$$

$$\frac{1}{2} = -\textcolor{blue}{\cos \frac{2\pi}{7}} + \cos \frac{3\pi}{7} + \textcolor{red}{\cos \frac{4\pi}{21}} + \cos \frac{10\pi}{21} \quad (\text{ix})$$

$$\frac{1}{2} = -\textcolor{blue}{\cos \frac{\pi}{15}} + \textcolor{red}{\cos \frac{2\pi}{15}} + \textcolor{red}{\cos \frac{4\pi}{15}} - \cos \frac{7\pi}{15} \quad (\text{x})$$

Notice that we need to include a rational sum of length at least three so that we have a cosine of the form $-\cos \frac{2\pi}{n}$, $n \geq 7$.

Sums of Length 3: If a solution contains a sum of length three then we are left with the remaining two cosines

$$\pm \cos a - \cos b = \frac{1}{2}. \quad (2.3.6)$$

Solutions to this are given when either one cosine is $\pm \frac{1}{2}$ and the other is 0 or we use sum (iii). Therefore we combine sums of length three with either of these possibilities to find suitable solutions for p , n , x and y .

The sum (iv) is discussed in proof of Proposition 2.3.0.23 and as no new angle arises, this sum produces no solution.

Consider the sum (v), if this is used for a solution then $\frac{2\pi}{n} = \frac{\pi}{15}$ and we must combine this sum with possible solutions to (2.3.6). In all cases there are only two possibilities for $\frac{3\pi}{p}$, either $\frac{3\pi}{p} = \frac{\pi}{5}$ or $\frac{3\pi}{p} = \frac{4\pi}{15}$.

The case $\frac{3\pi}{p} = \frac{\pi}{5}$ is discussed in Proposition 2.3.0.23 and produces no solutions.

For the case when $\frac{3\pi}{p} = \frac{4\pi}{15}$, one of the remaining three cosines must be equivalent to $\cos \frac{\pi}{5}$ and we have the following possibilities

$$\begin{aligned} 2 \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} &= -\cos x - \cos y - \cos z, \\ \cos \frac{\pi}{5} - \cos \frac{2\pi}{3} - \cos \frac{\pi}{2} &= -\cos x - \cos y - \cos z. \end{aligned}$$

Next we check whether $x + y + z = 2k\pi$ for $k \in \mathbb{Z}$. That is we search for solutions to

$$\begin{aligned} \frac{4\pi}{5} \pm \frac{4\pi}{5} \pm \frac{2\pi}{5} &= 2k\pi \\ \frac{4\pi}{5} \pm \frac{2\pi}{3} \pm \frac{\pi}{2} &= 2k\pi, \end{aligned}$$

respectively. The only possible solution is given by the first equation when we take the sign choices $(++)$. Therefore we have a solution to (2.3.3) when $\frac{3\pi}{p} = \frac{4\pi}{15}$ ($p = 11.25$), $\frac{2\pi}{n} = \frac{\pi}{15}$ ($n = 30$) and $\{x, y, z\} = \pm\{\frac{2\pi}{5}, \frac{4\pi}{5}, \frac{4\pi}{5}\}$. We will come back to this solution later on in the proof.

Sum (vi) cannot be used as there is no angle of the form $\frac{2\pi}{n}$.

Sums of Length 4: If we use a sum of length four then it must be combined with (ii) so that equation (2.3.3) is satisfied. As neither $\cos \frac{3\pi}{p}$ nor $-\cos \frac{2\pi}{n}$ is of the form of any of the cosines in (ii), the sum of length four must contain cosines of this type. For this reason we cannot use (vii) or (ix) as a solution, as none of the sums contain cosines of the form $\cos \frac{3\pi}{p}$ and $-\cos \frac{2\pi}{n}$ that satisfy $\frac{3\pi}{p} > \frac{2\pi}{n}$, with $p \geq 10$ and $n \geq 7$.

The sum (viii) is discussed in proof of Proposition 2.3.0.23 and as no new angle arises, this sum produces no solution.

Consider using (x) as a solution then it must be that $\frac{2\pi}{n} = \frac{\pi}{15}$ and we have two possibilities for $\frac{3\pi}{p}$, either $\frac{3\pi}{p} = \frac{2\pi}{15}$ or $\frac{3\pi}{p} = \frac{4\pi}{15}$. This leaves us with two cases to look at for the remaining three cosines, they are

$$\begin{aligned}\cos \frac{4\pi}{15} - \cos \frac{7\pi}{15} + \cos \frac{\pi}{3} &= -\cos x - \cos y - \cos z \\ \cos \frac{2\pi}{15} - \cos \frac{7\pi}{15} + \cos \frac{\pi}{3} &= -\cos x - \cos y - \cos z\end{aligned}$$

Again we search for solutions to

$$\begin{aligned}\frac{11\pi}{15} \pm \frac{7\pi}{15} \pm \frac{2\pi}{3} &= 2k\pi, \\ \frac{13\pi}{15} \pm \frac{7\pi}{15} \pm \frac{2\pi}{3} &= 2k\pi.\end{aligned}$$

Considering all possible sign combinations, the only solution is given by the second equation when we take the sign choices $(++)$. Therefore we have a solution to (2.3.3) again when $\frac{3\pi}{p} = \frac{4\pi}{15}$ ($p = 11.25$) and $\frac{2\pi}{n} = \frac{\pi}{15}$ ($n = 30$). But this time $\{x, y, z\} = \pm\{\frac{7\pi}{15}, \frac{2\pi}{3}, \frac{13\pi}{15}\}$.

Therefore, after checking all possible solutions to equation (2.3.3) we have suitable p, n, x, y and z that satisfies (2.3.3) for the triangle group $(15, 30, 30; 30)$ when $\{x, y, z\} = \pm\{\frac{14\pi}{15}, \frac{2\pi}{5}, \frac{2\pi}{3}\}$ and for the triangle group $(\frac{45}{4}, \frac{45}{2}, \frac{45}{2}; 30)$ when $\{x, y, z\} = \pm\{\frac{2\pi}{5}, \frac{4\pi}{5}, \frac{4\pi}{5}\}$ or $\{x, y, z\} = \pm\{\frac{7\pi}{15}, \frac{2\pi}{3}, \frac{13\pi}{15}\}$.

The triangle group $(15, 30, 30; 30)$ is discussed in the proof of Proposition 2.3.0.23 and is proven to be non-discrete. We now follow the same idea in checking the imaginary part of (2.1.2) also equates for the $(\frac{45}{4}, \frac{45}{2}, \frac{45}{2}; 30)$ triangle group. Fixing $p = 11.25$ and $n = 30$ determines the angular invariant α , which we find using (2.1.1) as

$$\alpha = 0.28726445111.$$

Now we need to check whether the imaginary part of (2.1.2) equates for the x, y

and z we found above. The imaginary part of the right hand side of (2.1.2) is

$$8r_1r_2r_3 \sin \alpha = 2.13663079881.$$

If we check the imaginary part of the left hand side for the possible x, y and z ;

$$\begin{aligned} \sin\left(\frac{2\pi}{5}\right) + \sin\left(\frac{4\pi}{5}\right) + \sin\left(\frac{4\pi}{5}\right) &= 2.12662702088 \\ \sin\left(\frac{7\pi}{15}\right) + \sin\left(\frac{2\pi}{3}\right) + \sin\left(\frac{13\pi}{15}\right) &= 2.26728394223 \end{aligned}$$

we find that the imaginary part of (2.1.2) does not equate, therefore the

$(\frac{45}{4}, \frac{45}{2}, \frac{45}{2}; 30)$ triangle group is not discrete. We can ignore the negative value as $8r_1r_2r_3 \sin \alpha > 0$.

□

Proposition 2.3.0.35. *The complex hyperbolic triangle group $\Gamma(p, 3p/2, 3p; n)$ is non-discrete for all $p \in \mathbb{Q}$, $p \geq 11$ and w_B is regular elliptic.*

Remark 2.3.0.36. *The fact we consider the triangle group $\Gamma(p, 3p/2, 3p; n)$ with n finite implies that w_A is elliptic.*

Proof. In the $(p, 3p/2, 3p)$ case, equation (2.3.2) becomes

$$1 = \cos \frac{10\pi}{3p} - \cos x - \cos y - \cos z - \cos \frac{2\pi}{n}, \quad (2.3.5)$$

with the conditions $p \geq 11$, $7 \leq n \in \mathbb{Z}$, $\frac{10\pi}{3p} > \frac{2\pi}{n}$ and $x + y + z = 2k\pi$. We need to find solutions to equation (2.3.5) using the set **A**. Taking care again that our equation can have subsets which are rational, the angles are not necessarily distinct and do not necessarily lie in $(0, \frac{\pi}{2})$.

We procede as in the previous proofs and we mark out the cosines on our list that can be of the form $\cos \frac{10\pi}{3p}$, $p \geq 11$, in red and those of the form $-\cos \frac{2\pi}{n}$,

$n \geq 7$, in blue,

$$\frac{1}{2} = \cos \frac{\pi}{3} \quad (\text{ii})$$

$$\frac{1}{2} = \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} \quad (\text{iii})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} \quad (\text{iv})$$

$$\frac{1}{2} = \cos \frac{\pi}{5} - \cos \frac{\pi}{15} + \cos \frac{4\pi}{15} \quad (\text{v})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{5} + \cos \frac{2\pi}{15} - \cos \frac{7\pi}{15} \quad (\text{vi})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{2\pi}{21} - \cos \frac{5\pi}{21} \quad (\text{vii})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} + \cos \frac{3\pi}{7} - \cos \frac{\pi}{21} + \cos \frac{8\pi}{21} \quad (\text{viii})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{4\pi}{21} + \cos \frac{10\pi}{21} \quad (\text{ix})$$

$$\frac{1}{2} = -\cos \frac{\pi}{15} + \cos \frac{2\pi}{15} + \cos \frac{4\pi}{15} - \cos \frac{7\pi}{15} \quad (\text{x})$$

Notice that this is exactly the same set of cosines we had in the proof of Proposition 2.3.0.33, hence if we follow the exact same proof it will lead to the same solutions that we found for the $(p, 2p, 2p; n)$ triangle groups.

That is we have a solution when $\frac{10\pi}{3p} = \frac{\pi}{5}$ ($p = \frac{50}{3}$), $\frac{2\pi}{n} = \frac{\pi}{15}$ ($n = 30$) and $\{x, y, z\} = \pm\{\frac{14\pi}{15}, \frac{2\pi}{5}, \frac{2\pi}{3}\}$ and we have a solution when $\frac{10\pi}{3p} = \frac{4\pi}{15}$ ($p = \frac{25}{2}$), $\frac{2\pi}{n} = \frac{\pi}{15}$ ($n = 30$) and $\{x, y, z\} = \pm\{\frac{2\pi}{5}, \frac{4\pi}{5}, \frac{4\pi}{5}\}$ or $\{x, y, z\} = \pm\{\frac{7\pi}{15}, \frac{2\pi}{3}, \frac{13\pi}{15}\}$.

Therefore we just need to check whether the imaginary part of (2.1.2) equates or not for the given p, n, x, y and z .

For the $(\frac{50}{3}, 25, 50; 30)$ triangle group we have the angular invariant α , found using (2.1.1) as

$$\alpha = 0.2088813423$$

We then check the imaginary part of (2.1.2):

$$\begin{aligned} \text{RHS:} & \quad 8r_1r_2r_3 \sin \alpha = 1.613501778 \\ \text{LHS:} & \quad \sin\left(\frac{14\pi}{15}\right) + \sin\left(\frac{2\pi}{5}\right) + \sin\left(\frac{2\pi}{3}\right) = 2.0249993611, \end{aligned}$$

Hence the $(\frac{50}{3}, 25, 50; 30)$ triangle group is not discrete. For the $(\frac{25}{2}, \frac{75}{4}, \frac{75}{2}; 30)$ triangle group we have the angular invariant α , again found using (2.1.1) as

$$\alpha = 0.28588940517$$

We compute the imaginary part of (2.1.2) as

$$\begin{aligned} \text{RHS:} & \quad 8r_1r_2r_3 \sin \alpha = 2.14704974737 \\ \text{LHS:} & \quad \sin\left(\frac{2\pi}{5}\right) + \sin\left(\frac{4\pi}{5}\right) + \sin\left(\frac{4\pi}{5}\right) = 2.12662702088 \\ \text{or} & \quad \sin\left(\frac{7\pi}{15}\right) + \sin\left(\frac{2\pi}{3}\right) + \sin\left(\frac{13\pi}{15}\right) = 2.26728394223 \end{aligned}$$

Clearly, neither equate so the $(\frac{25}{2}, \frac{75}{4}, \frac{75}{2}; 30)$ triangle group is not discrete. Therefore all $(p, 3p/2, 3p)$ triangle groups with $p \geq 11$ and w_A and w_B both elliptic are not discrete.

□

Remark 2.3.0.37. *We use the same sums to obtain a possible solution in both of the above proofs. This is because no new angles arise from $(p, 2p, 2p)$ to $(p, 3p/2, 3p)$ that can be written in the form $\frac{2\pi}{n}$ or $(2\varphi_1 + 2\varphi_2)$. Therefore we will try to generalise this result for all (p_1, p_2, p_3) triangle groups of the form $\varphi_1 = \varphi_2 + \varphi_3$.*

Theorem 2.3.0.38. *The complex hyperbolic triangle group $\Gamma(p_1, p_2, p_3; n)$ with $\frac{\pi}{p_1} = \frac{\pi}{p_2} + \frac{\pi}{p_3}$ is non-discrete for all $p_1 \in \mathbb{Z}$, $p_1 \geq 12$ when w_B is regular elliptic.*

Remark 2.3.0.39. *The fact we consider the triangle group $\Gamma(p_1, p_2, p_3; n)$ with n finite implies that w_A is elliptic.*

Proof. We can write the triangle group (p_1, p_2, p_3) with $\frac{\pi}{p_1} = \frac{\pi}{p_2} + \frac{\pi}{p_3}$ as a $(p, \frac{ap}{b}, \frac{ap}{a-b})$ triangle group, where $a, b \in \mathbb{N}$, $a > b$, $a \geq 2$, $\gcd(a, b) = 1$ and $a \leq 2b$.

Remark 2.3.0.40. *The conditions $a > b$ and $a \leq 2b$ just order the angles of the triangle so that $p_1 \leq p_2 \leq p_3$.*

We can then rewrite equation 2.3.2 as,

$$1 = \cos \left(\frac{2(a+b)\pi}{ap} \right) - \cos x - \cos y - \cos z - \cos \frac{2\pi}{n}.$$

Now,

$$\frac{2(a+b)\pi}{ap} = \left(2 + \frac{2b}{a} \right) \frac{\pi}{p} < \frac{4\pi}{p},$$

For $p \geq 12$ we have that

$$\frac{2\pi}{n} < \frac{2(a+b)\pi}{ap} < \frac{\pi}{3},$$

which is the same situation we had in the cases $(p, 2p, 2p)$ and $(p, 3p/2, 3p)$, already discussed. Hence the only solutions occur when,

$$\begin{aligned} \frac{2\pi}{n} = \frac{\pi}{15} &\Leftrightarrow n = 30, \\ \text{and} \quad \frac{2(a+b)\pi}{ap} = \frac{\pi}{5} &\Leftrightarrow p = \frac{10(a+b)}{a}, \end{aligned}$$

or

$$\begin{aligned} \frac{2\pi}{n} = \frac{\pi}{15} &\Leftrightarrow n = 30, \\ \text{and} \quad \frac{2(a+b)\pi}{ap} = \frac{4\pi}{15} &\Leftrightarrow p = \frac{15(a+b)}{2a}. \end{aligned}$$

First consider the case

$$p = \frac{10(a+b)}{a}.$$

For p to be an integer we need

$$a \mid 10(a+b).$$

Suppose a shares factors with 10 and $(a+b)$, that is we can write $a = cd$, where c divides 10 and d divides $(a+b)$. So $d \mid a$ and $d \mid (a+b)$, hence $d \mid b$, but $\gcd(a, b) = 1$, thus $d = 1$. Hence a must divide 10 only.

We have the following possible solutions that satisfy all the conditions, they are:

- $a = 2$ and $b = 1$ which corresponds to a $(15, 30, 30; 30)$ triangle group
- $a = 5$ and $b = 3$ which corresponds to a $(16, \frac{80}{3}, 40; 30)$ triangle group
- $a = 10$ and $b = 7$ which corresponds to a $(17, \frac{170}{7}, \frac{170}{3}; 30)$ triangle group
- $a = 5$ and $b = 4$ which corresponds to a $(18, \frac{45}{2}, 90; 30)$ triangle group
- $a = 10$ and $b = 9$ which corresponds to a $(19, \frac{190}{9}, 190; 30)$ triangle group

We now check for each triangle group whether the imaginary part of (2.1.2) equates. In each case we know that the LHS of the equation is

$$\sin\left(\frac{14\pi}{15}\right) + \sin\left(\frac{2\pi}{5}\right) + \sin\left(\frac{2\pi}{3}\right) = 2.0249993611.$$

For each of the triangle groups we have a fixed p and fixed n which determines an angular invariant α which we find in each case and then compute the RHS of equation (2.1.2).

- $(15, 30, 30; 30)$:

$$\begin{aligned}\alpha &= 0.2094395099 \\ 8r_1r_2r_3 \sin \alpha &= 1.609170235.\end{aligned}$$

- $(16, \frac{80}{3}, 40; 30)$:

$$\begin{aligned}\alpha &= 0.2090576819 \\ 8r_1r_2r_3 \sin \alpha &= 1.612130849.\end{aligned}$$

- $(17, \frac{170}{7}, \frac{170}{3}; 30)$:

$$\begin{aligned}\alpha &= 0.2088119868 \\ 8r_1r_2r_3 \sin \alpha &= 1.614041612.\end{aligned}$$

- $(18, \frac{45}{2}, 90; 30)$:

$$\begin{aligned}\alpha &= 0.2086649472 \\ 8r_1r_2r_3 \sin \alpha &= 1.615187281.\end{aligned}$$

- $(19, \frac{190}{9}, 190; 30)$:

$$\begin{aligned}\alpha &= 0.2085901051 \\ 8r_1r_2r_3 \sin \alpha &= 1.615771035.\end{aligned}$$

Clearly none of these equate. Next consider the case

$$p = \frac{15(a+b)}{2a}.$$

For p to be an integer we need

$$2|(a+b) \quad \text{and} \quad a \mid 15 \frac{(a+b)}{2}$$

Now suppose a shares factors with 15 and $\frac{(a+b)}{2}$, that is we can write $a = cd$, where c divides 15 and d divides $\frac{(a+b)}{2}$. If d divides $\frac{(a+b)}{2}$, d certainly divides $(a+b)$. So $d|a$ and $d|(a+b)$, hence $d|b$, but $\gcd(a, b) = 1$, thus $d = 1$. Hence a must divide 15 only.

We have only three possible solutions that satisfy all the conditions, they are:

- $a = 5$ and $b = 3$ which corresponds to a $(12, 20, 30; 30)$ triangle group
- $a = 15$ and $b = 11$ which corresponds to a $(13, \frac{195}{11}, \frac{195}{4}; 30)$ triangle group
- $a = 15$ and $b = 13$ which corresponds to a $(14, \frac{210}{13}, 105; 30)$ triangle group

Again we check for each triangle group whether the imaginary part of (2.1.2) equates. In each case we know that the LHS of the equation is

$$\begin{aligned} \sin\left(\frac{2\pi}{5}\right) + \sin\left(\frac{4\pi}{5}\right) + \sin\left(\frac{4\pi}{5}\right) &= 2.12662702088 \\ \text{or} \quad \sin\left(\frac{7\pi}{15}\right) + \sin\left(\frac{2\pi}{3}\right) + \sin\left(\frac{13\pi}{15}\right) &= 2.26728394223 \end{aligned}$$

For each of the triangle groups we have a fixed p and fixed n which determines an angular invariant α which we find in each case and then compute the RHS of equation (2.1.2).

- $(12, 20, 30; 30)$:

$$\alpha = 0.2863232857$$

$$8r_1r_2r_3 \sin \alpha = 2.14375142569.$$

- $(13, \frac{195}{11}, \frac{195}{4}; 30)$:

$$\alpha = 0.2855751654$$

$$8r_1r_2r_3 \sin \alpha = 2.149444783883.$$

- $(14, \frac{210}{13}, 105; 30)$:

$$\alpha = 0.2852186171$$

$$8r_1r_2r_3 \sin \alpha = 2.15216863477.$$

Thus none of these three triangle groups are discrete and hence every complex hyperbolic triangle group $\Gamma(p_1, p_2, p_3; n)$ with $\frac{\pi}{p_1} = \frac{\pi}{p_2} + \frac{\pi}{p_3}$ is non-discrete for all $p_1 \in \mathbb{Z}$, with $p_1 \geq 12$ when $\Gamma(p_1, p_2, p_3)$ is of type B and w_B is elliptic regular elliptic.

□

If we consider a general (p_1, p_2, p_3) triangle group then equation (2.1.3) will be a rational linear combination of seven cosines of rational multiples of π with a rational sum. These length of sums are not covered in the works of Conway and Jones or Parker. Therefore if we wish to study solutions to these we need to extend their results. In the next section we do exactly this.

2.4 Vanishing Sums of Roots of Unity

In the following sections we will look at solutions to the equation

$$A_0 = \sum_{i=1}^7 A_i \cos(\pi a_i), \quad \text{where } A_i, a_i \in \mathbb{Q}. \quad (2.4.1)$$

Sums similar to this form were discussed by Conway and Jones [2] and John Parker [21], and we shall use their approach. The question of finding such sums is reduced to that of identifying linear dependences between roots of unity. To do this they define vanishing sums of rational multiples of roots of unity where there is no additive cancellation and find criteria which enable them to find all vanishing sums of a given length. Then by looking at the real parts of vanishing sums they obtain vanishing rational linear combinations of cosines of rational multiples of π . This theory is used in Theorem 7 of [2] to compute all rational linear combinations of four cosines of rational multiples of π which equal a rational number and used in Theorem A.3.1 of [21] to determine all vanishing rational linear combinations of six cosines of rational multiples of π .

The following chapter will be dedicated to proving Theorem 2.4.3.1 so that we can find a complete solution to equation (2.4.1). First we will give some of the results from [2].

2.4.1 The Results of Conway and Jones

We shall now present the theory of formal sums of roots of unity introduced by Conway and Jones [2].

Definition. *Define the collection of roots of unity as $\mathcal{R} = \{\exp(2\pi q) : q \in \mathbb{Q}\}$ and denote the space of maps $f : \mathcal{R} \rightarrow \mathbb{Q}$ by \mathcal{V} . For each root of unity $\alpha \in \mathcal{R}$ we define a map $L_\alpha \in \mathcal{V}$ by $L_\alpha : \alpha \mapsto 1$ and $L_\alpha : \beta \mapsto 0$ for $\beta \neq \alpha$. Therefore L_α is the delta function at α .*

The space \mathcal{V} inherits a vector space structure from \mathbb{Q} which gives the natural addition on $L, K \in \mathcal{V}$ by:

$$(L + K) : \alpha \mapsto L(\alpha) + K(\alpha).$$

If we try to define multiplication in a similar way, then for any two distinct

roots α and β we would have $L_\alpha L_\beta \equiv 0$. Therefore we define multiplication by $L_\alpha L_\beta = L_\gamma$, where $\gamma = \alpha\beta$. Thus we have an addition and a multiplication on \mathcal{V} .

Definition. Define the ring of formal sums of roots of unity \mathcal{F} to be all functions $\mathbf{S} \in \mathcal{V}$ of the form $\mathbf{S} = \sum c_\alpha L_\alpha$, where L_α ranges over all $\alpha \in \mathcal{R}$ and $c_\alpha \in \mathbb{Q}$ with $c_\alpha = 0$ for all but finitely many $\alpha \in \mathcal{R}$, with the addition and multiplication defined above.

Definition. The formal sum \mathbf{S} is said to involve α when the expression for \mathbf{S} in terms of the above basis has non-zero coefficient of L_α .

Definition. The length $l(\mathbf{S})$, of \mathbf{S} is the number of roots involved in \mathbf{S} .

Definition. The sum \mathbf{S} is called similar to $kL_\alpha \mathbf{S}$ for any root α and any $k \in \mathbb{Q} \setminus \{0\}$.

Definition. The exponent $e(\mathbf{S})$, of \mathbf{S} is the lowest common multiple of the orders of the roots involved in \mathbf{S} .

Definition. The reduced exponent $r(\mathbf{S})$, of \mathbf{S} is the smallest exponent of any sum similar to \mathbf{S} .

There is a vector space homomorphism v from \mathcal{F} to \mathbb{C} given by evaluation, that is $v : L_\alpha \mapsto \alpha$ for all roots $\alpha \in \mathcal{R}$.

Definition. The value $v(\mathbf{S})$, of the sum $\mathbf{S} = \sum c_\alpha L_\alpha$ is the complex number $\sum c_\alpha \alpha$.

Definition. We say that the sum \mathbf{S} is a vanishing sum if it is in the kernel of v , that is $v(\mathbf{S}) = 0$.

Definition. A vanishing sum \mathbf{S} is said to be minimal if no proper subsum vanishes.

Definition. If the sum \mathbf{S} involves 1 we call \mathbf{S} monic.

Suppose \mathbf{S} is monic, that is we can write

$$\mathbf{S} = c_1 L_1 + c_\alpha L_\alpha + c_\beta L_\beta + \dots$$

where α and β are roots of unity. Now the exponent of \mathbf{S} is

$$e(\mathbf{S}) = \text{lcm}(\text{ord}(\alpha), \text{ord}(\beta), \dots).$$

If we take any sum similar to \mathbf{S} , say $kL_\omega \mathbf{S}$ then

$$\begin{aligned} e(kL_\omega \mathbf{S}) &= \text{lcm}(\text{ord}(\omega), \text{ord}(\omega\alpha), \text{ord}(\omega\beta), \dots) \\ &= \text{lcm}(\text{ord}(\omega), \text{lcm}(\text{ord}(\omega), \text{ord}(\alpha)), \text{lcm}(\text{ord}(\omega), \text{ord}(\beta)), \dots) \\ &= \text{lcm}(\text{ord}(\omega), \text{ord}(\alpha), \text{ord}(\beta), \dots) \\ &= \text{lcm}(\text{ord}(\omega), e(\mathbf{S})) \\ &\geq e(\mathbf{S}). \end{aligned}$$

Therefore if \mathbf{S} is monic then the exponent coincides with its reduced exponent, i.e. $e(\mathbf{S}) = r(\mathbf{S})$. We can now give some properties of formal sums.

Proposition 2.4.1.1. *Let \mathbf{S} be a formal sum of roots of unity. If \mathbf{S} is a minimal vanishing sum then $r(\mathbf{S})$ is square-free.*

Proof. See Theorem 1 of [2].

□

Suppose that $\mathbf{S} = \sum c_\alpha L_\alpha$ is a minimal vanishing sum, then any sum similar to \mathbf{S} is also a minimal vanishing sum. Therefore we may assume that $e(\mathbf{S}) = r(\mathbf{S})$. Hence from Proposition 2.4.1.1 we may assume that every root α involved in \mathbf{S} has square-free order. Now suppose that we have a root β involved in \mathbf{S} which has even order. This implies that the order of β is $2n$, where n is odd, since if n was even then β would not be of square-free order. Then the root $-\beta$ has odd

order, n . Therefore we define a projection map $\Pi : \mathcal{F} \rightarrow \mathcal{F}$ given by:

$$\begin{aligned} \Pi & : c_\alpha L_\alpha + c_{-\alpha} L_{-\alpha} \mapsto (c_\alpha - c_{-\alpha}) L_\alpha, & \text{if } \alpha \text{ is of odd order} \\ \text{and } \Pi & : c_\alpha L_\alpha + c_{-\alpha} L_{-\alpha} \mapsto (-c_\alpha + c_{-\alpha}) L_{-\alpha}, & \text{if } \alpha \text{ is of even order.} \end{aligned}$$

for any root α . Thus $\Pi\mathbf{S}$ contains only roots of odd order and so $e(\Pi\mathbf{S})$ is odd.

Remark 2.4.1.2. *The projection Π does not alter the value of the sum, that is $v(\Pi\mathbf{S}) = v(\mathbf{S})$. Therefore without loss of generality we only consider minimal vanishing sums that are in $\Pi\mathcal{F}$. Such sums have odd, square-free reduced exponent.*

Example. *Consider the vanishing sum $\mathbf{S} = L_{\omega^5} + L_\omega - L_1$, where $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ is a primitive sixth root of unity. As ω has even order, when we apply the projection Π we replace L_ω by $-(L_{-\omega})$. We can now use that $-\omega = \omega^4$, which has order 3, to replace $L_{-\omega}$ by L_{ω^4} . Similarly we replace L_{ω^5} by $-(L_{-\omega^5}) = -L_{\omega^2}$ and obtain $\Pi\mathbf{S} = -L_{\omega^2} - L_{\omega^4} - L_1$, which involves only roots of odd order.*

Proposition 2.4.1.3. *Let ω be a primitive p^{th} root of unity, where p is prime and suppose*

$$\mathbf{S} = L_1 \mathbf{S}_0 + L_\omega \mathbf{S}_1 + L_{\omega^2} \mathbf{S}_2 + \dots + L_{\omega^{p-1}} \mathbf{S}_{p-1}$$

is a vanishing sum, with each \mathbf{S}_i having exponents prime to p . Then \mathbf{S}_i all have the same value.

Proof. See Lemma 1 of [2].

□

Proposition 2.4.1.4. *Let \mathbf{S} be a minimal vanishing sum with length l and reduced exponent r . Then*

$$l \geq 2 + \sum_{p|r} (p - 2),$$

the sum is taken over all prime p such that $p|r$.

Proof. See Theorem 5 of [2].

□

2.4.2 Vanishing Sums of Small Length

To prove Theorem 2.4.3.1 we first prove the following theorem, which is an extension of Theorem 6 of [2] and Theorem A.3.1 of [21].

Notation. From this point on we shall denote the map $L_\alpha \in \mathcal{V}$ by α .

Theorem 2.4.2.1. Let $\alpha, \beta, \gamma, \delta$ and ξ be primitive roots of unity of orders 3, 5, 7, 11 and 13 respectively. Let \mathbf{S} be a non-empty vanishing sum of length at most 15. Then one of the following is true:

a) For some root of unity θ , the sum \mathbf{S} involves one of the five sets of roots of unity:

$$\begin{aligned} &\{\theta, \alpha\theta, \alpha^2\theta\}, \quad \{\theta, \beta\theta, \beta^2\theta, \beta^3\theta, \beta^4\theta\}, \quad \{\alpha\theta, \alpha^2\theta, \beta\theta, \beta^2\theta, \beta^3\theta, \beta^4\theta\}, \\ &\{\theta, \gamma\theta, \gamma^2\theta, \gamma^3\theta, \gamma^4\theta, \gamma^5\theta, \gamma^6\theta\}, \quad \{\theta, \theta\beta^2, \theta\beta^3, \theta\alpha\beta, \theta\alpha\beta^4, \theta\alpha^2\beta, \theta\alpha^2\beta^4\} \end{aligned}$$

b) The sum \mathbf{S} is similar to one of the following sums:

$$\begin{aligned}
& 1 + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 - (\alpha + \alpha^2)(\gamma + \gamma^6) \\
& -(\beta + \beta^2 + \beta^3 + \beta^4) + \gamma + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 + \gamma^6 \\
& 1 + \delta + \delta^2 + \delta^3 + \delta^4 + \delta^5 + \delta^6 + \delta^7 + \delta^8 + \delta^9 + \delta^{10} \\
& 1 + \gamma + \gamma^6 - (\alpha + \alpha^2)(\gamma^2 + \gamma^3 + \gamma^4 + \gamma^5) \\
& -(\alpha + \alpha^2) + \delta + \delta^2 + \delta^3 + \delta^4 + \delta^5 + \delta^6 + \delta^7 + \delta^8 + \delta^9 + \delta^{10} \\
& -(\beta + \beta^2 + \beta^3 + \beta^4) + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 - (\alpha + \alpha^2)(\gamma + \gamma^6) \\
& (\alpha + \alpha^2)(\beta + \beta^4) - (\beta^2 + \beta^3) + \gamma + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 + \gamma^6 \\
& 1 - (\alpha + \alpha^2)(\gamma + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 + \gamma^6) \\
& 1 + \xi + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \xi^6 + \xi^7 + \xi^8 + \xi^9 + \xi^{10} + \xi^{11} + \xi^{12} \\
& 1 + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 - (\beta + \beta^2 + \beta^3 + \beta^4)(\gamma + \gamma^6) \\
& 1 + \delta^2 + \delta^3 + \delta^4 + \delta^5 + \delta^6 + \delta^7 + \delta^8 + \delta^9 - (\alpha + \alpha^2)(\delta + \delta^{10}) \\
& -(\beta + \beta^2 + \beta^3 + \beta^4) + \gamma + \gamma^6 - (\alpha + \alpha^2)(\gamma^2 + \gamma^3 + \gamma^4 + \gamma^5) \\
& -(\alpha + \alpha^2) + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 - (\beta + \beta^2 + \beta^3 + \beta^4)(\gamma + \gamma^6) \\
& (\alpha + \alpha^2)(\beta + \beta^2 + \beta^3 + \beta^4) + \gamma + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 + \gamma^6 \\
& -(\alpha + \alpha^2)(1 + \delta + \delta^{10}) + \delta^2 + \delta^3 + \delta^4 + \delta^5 + \delta^6 + \delta^7 + \delta^8 + \delta^9 \\
& -(\beta + \beta^2 + \beta^3 + \beta^4) + \delta + \delta^2 + \delta^3 + \delta^4 + \delta^5 + \delta^6 + \delta^7 + \delta^8 + \delta^9 + \delta^{10} \\
& -(\alpha + \alpha^2) + \xi + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \xi^6 + \xi^7 + \xi^8 + \xi^9 + \xi^{10} + \xi^{11} + \xi^{12} \\
& 1 + \beta^2 + \beta^3 - (\beta + \beta^4)(\gamma + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 + \gamma^6) \\
& 1 - (\alpha + \alpha^2)(\gamma + \gamma^6) - (\beta + \beta^2 + \beta^3 + \beta^4)(\gamma^2 + \gamma^5) + \gamma^3 + \gamma^4 \\
& 1 - (\alpha + \alpha^2)(\gamma + \gamma^6) - (\beta + \beta^2 + \beta^3 + \beta^4)(\gamma^3 + \gamma^4) + \gamma^2 + \gamma^5 \\
& 1 + \delta^3 + \delta^4 + \delta^5 + \delta^6 + \delta^7 + \delta^8 - (\alpha + \alpha^2)(\delta + \delta^2 + \delta^9 + \delta^{10}) \\
& 1 + \delta^2 + \delta^4 + \delta^5 + \delta^6 + \delta^7 + \delta^9 - (\alpha + \alpha^2)(\delta + \delta^3 + \delta^8 + \delta^{10}) \\
& 1 + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \xi^6 + \xi^7 + \xi^8 + \xi^9 + \xi^{10} + \xi^{11} - (\alpha + \alpha^2)(\xi + \xi^{12}).
\end{aligned}$$

Proof. First suppose that \mathbf{S} is minimal. Then \mathbf{S} is similar to a minimal vanishing sum with $e(\mathbf{S}) = r(\mathbf{S})$. In this case we can assume that its reduced exponent is square-free and odd, by Proposition 2.4.1.1 and the ensuing observation. Let p be the largest prime divisor of $r(\mathbf{S})$ so that by Proposition 2.4.1.4, $p \leq 13$. Let ω be a primitive p^{th} root of unity and express \mathbf{S} as,

$$\mathbf{S}_0 + \omega \mathbf{S}_1 + \dots + \omega^{p-1} \mathbf{S}_{p-1},$$

where \mathbf{S}_i have exponents prime to p and remembering that $\omega = L_\omega$.

If $p = 3$, then \mathbf{S} can involve only $1, \alpha, \alpha^2$ and so it must be that \mathbf{S} is similar to $\mathbf{S} = \mathbf{1} + \alpha + \alpha^2$.

If $p \geq 5$, there are two cases, either all of the \mathbf{S}_i have more than one term or there is at least one \mathbf{S}_i with length one.

Let us first assume that all of the \mathbf{S}_i have more than one term. If $p = 5$ we have $\mathbf{S} = \mathbf{S}_0 + \beta \mathbf{S}_1 + \dots + \beta^4 \mathbf{S}_4$, therefore either all \mathbf{S}_i have length three or there is an \mathbf{S}_i with length two, otherwise $l(\mathbf{S}) > 15$. Since \mathbf{S}_i all have the same value, by Proposition 2.4.1.3, no \mathbf{S}_i has value zero otherwise \mathbf{S} is not minimal. Therefore we have five non-vanishing formal sums \mathbf{S}_i , all of length two or three, only involving $1, \alpha, \alpha^2$.

If one of $1, \alpha, \alpha^2$ is involved in all five sums then \mathbf{S} involves $\{\theta, \beta\theta, \beta^2\theta, \beta^3\theta, \beta^4\theta\}$ for some root θ . If none of $1, \alpha, \alpha^2$ is involved in all five sums, then at least one is involved in four of them. By multiplying by the appropriate root of unity, we may assume that 1 is involved in $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ and \mathbf{S}_4 . Then α and α^2 are involved in \mathbf{S}_0 and hence \mathbf{S} involves $\{\alpha\theta, \alpha^2\theta, \beta\theta, \beta^2\theta, \beta^3\theta, \beta^4\theta\}$ for some root θ .

If $p = 7$ and all \mathbf{S}_i have more than one term, then either all \mathbf{S}_i have length two or there is one \mathbf{S}_i with length three and the rest have length two, otherwise $l(\mathbf{S}) > 15$. Since the \mathbf{S}_i all have the same value, by Proposition 2.4.1.3, no \mathbf{S}_i has value zero otherwise \mathbf{S} is not minimal. If some $\mathbf{S}_i \neq \mathbf{S}_j$, then $\mathbf{S}_i - \mathbf{S}_j$ is a vanishing sum with exponent prime to p with $l(\mathbf{S}_i - \mathbf{S}_j) \leq 5$. If $l(\mathbf{S}_i - \mathbf{S}_j) \leq 4$

then \mathbf{S} involves $\{\theta, \gamma\theta, \gamma^2\theta, \gamma^3\theta, \gamma^4\theta, \gamma^5\theta, \gamma^6\theta\}$, for some root θ . If $l(\mathbf{S}_i - \mathbf{S}_j) = 5$ then $\mathbf{S}_i - \mathbf{S}_j$ is

$$k\varepsilon(1 + \beta + \beta^2 + \beta^3 + \beta^4),$$

for some $k \in \mathbb{Q}$ and a root of unity ε . Suppose $k\varepsilon \neq 1$, then \mathbf{S} would involve $\varepsilon\omega^i, \beta\varepsilon\omega^i, \beta^2\varepsilon\omega^i, \beta^3\varepsilon\omega^i, \beta^4\varepsilon\omega^i$ i.e. \mathbf{S} involves $\{\theta, \beta\theta, \beta^2\theta, \beta^3\theta, \beta^4\theta\}$ for $\theta = \varepsilon\omega^i$. If $k\varepsilon = 1$ then \mathbf{S} is similar to

$$1 + \beta^2 + \beta^3 - (\beta + \beta^4)(\gamma + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 + \gamma^6).$$

If $p \geq 11$ and all of the \mathbf{S}_i have more than one term, then $l(\mathbf{S}) \geq 2p \geq 22$. Therefore it must be that some \mathbf{S}_i has at most one term. We may now therefore assume that there is at least one \mathbf{S}_i with length one. Since \mathbf{S}_i all have the same value, by Proposition 2.4.1.3, no \mathbf{S}_i has value zero otherwise \mathbf{S} is not minimal and, without loss of generality, we may assume that $\mathbf{S}_0 = \mathbf{1}$ and hence all the \mathbf{S}_i have value 1. If some $\mathbf{S}_i \neq \mathbf{1}$, then $\mathbf{1} - \mathbf{S}_i$ is a vanishing sum with reduced exponent prime to p . Note that all \mathbf{S}_i are of length at least one, hence for each \mathbf{S}_i we have

$$l(\mathbf{S}_i) \leq l(\mathbf{S}) - (p - 1) \leq 16 - p.$$

Therefore

$$\begin{aligned} l(\mathbf{1} - \mathbf{S}_i) &\leq 16 - p + 1, & \text{if } \mathbf{S}_i \text{ is not monic and hence } \mathbf{1} - \mathbf{S}_i \text{ is monic} \\ \text{and } l(\mathbf{1} - \mathbf{S}_i) &\leq 16 - p - 1, & \text{if } \mathbf{S}_i \text{ is monic and hence } \mathbf{1} - \mathbf{S}_i \text{ is not monic.} \end{aligned}$$

If $p = 5$ then it is clear that $r(\mathbf{1} - \mathbf{S}_i) = 3$. Therefore $\mathbf{1} - \mathbf{S}_i$ is

$$k\varepsilon(1 + \alpha + \alpha^2),$$

for some $k \in \mathbb{Q}$ and a root of unity ε . If $p = 7$, then $l(\mathbf{1} - \mathbf{S}_i) \leq 10$ if $\mathbf{1} - \mathbf{S}_i$ is monic and $l(\mathbf{1} - \mathbf{S}_i) \leq 8$ if $\mathbf{1} - \mathbf{S}_i$ is not monic with $r(\mathbf{1} - \mathbf{S}_i) = 3, 5$ or 15 . Therefore $\mathbf{1} - \mathbf{S}_i$ is one of the following sums:

$$\begin{aligned} & k\varepsilon(\mathbf{1} + \alpha + \alpha^2) \\ & k\varepsilon(\mathbf{1} + \beta + \beta^2 + \beta^3 + \beta^4) \\ \text{or} \quad & k\varepsilon(\mathbf{1} + \beta^2 + \beta^3 - (\alpha + \alpha^2)(\beta + \beta^4)) \end{aligned}$$

for some $k \in \mathbb{Q}$ and a root of unity ε . Otherwise \mathbf{S} involves $\{\theta, \alpha\theta, \alpha^2\theta\}$, $\{\theta, \beta\theta, \beta^2\theta, \beta^3\theta, \beta^4\theta\}$ or $\{\alpha\theta, \alpha^2\theta, \beta\theta, \beta^2\theta, \beta^3\theta, \beta^4\theta\}$ for some root θ .

If $p = 11$, then $l(\mathbf{1} - \mathbf{S}_i) \leq 6$ if $\mathbf{1} - \mathbf{S}_i$ is monic and $l(\mathbf{1} - \mathbf{S}_i) \leq 4$ if $\mathbf{1} - \mathbf{S}_i$ is not monic. Vanishing sums of length less than or equal to six are covered by Conway and Jones and we see that $\mathbf{1} - \mathbf{S}_i$ is one of the following sums:

$$\begin{aligned} & k\varepsilon(\mathbf{1} + \alpha + \alpha^2) \\ \text{or} \quad & k\varepsilon(\mathbf{1} + \beta + \beta^2 + \beta^3 + \beta^4) \end{aligned}$$

for some $k \in \mathbb{Q}$ and a root of unity ε . If $p = 13$, then $l(\mathbf{1} - \mathbf{S}_i) \leq 4$ if $\mathbf{1} - \mathbf{S}_i$ is monic and $l(\mathbf{1} - \mathbf{S}_i) \leq 2$ if $\mathbf{1} - \mathbf{S}_i$ is not monic. Vanishing sums of length less than or equal to four are covered by Conway and Jones and we see that $\mathbf{1} - \mathbf{S}_i$ must be

$$k\varepsilon(\mathbf{1} + \alpha + \alpha^2)$$

for some $k \in \mathbb{Q}$ and a root of unity ε .

Therefore $\mathbf{1} - \mathbf{S}_i$ can only be one of the following sums:

$$\begin{aligned} & k\varepsilon(\mathbf{1} + \alpha + \alpha^2) \\ & k\varepsilon(\mathbf{1} + \beta + \beta^2 + \beta^3 + \beta^4) \\ \text{or} \quad & k\varepsilon(\mathbf{1} + \beta^2 + \beta^3 - (\alpha + \alpha^2)(\beta + \beta^4)) \end{aligned}$$

for some $k \in \mathbb{Q}$ and a root of unity ε . Suppose $\mathbf{1} - \mathbf{S}_i = k\varepsilon(\mathbf{1} + \alpha + \alpha^2)$ and $k\varepsilon \neq 1$, then \mathbf{S} would involve $\varepsilon\omega^i, \alpha\varepsilon\omega^i, \alpha^2\varepsilon\omega^i$ i.e. \mathbf{S} involves $\{\theta, \alpha\theta, \alpha^2\theta\}$ for $\theta = \varepsilon\omega^i$. A similar argument can be used if $\mathbf{1} - \mathbf{S}_i$ is equal to one of the other two equations, so that if $k\varepsilon \neq 1$ then \mathbf{S} is covered in part a) of the theorem.

If $k\varepsilon = 1$, we substitute our possible \mathbf{S}_i into the sum \mathbf{S} for $p = 5, 7, 11$ and $p = 13$ and obtain sums similar to those given in part b) of the theorem. For example, when $p = 13$ the sum \mathbf{S} is of the form

$$\mathbf{S} = \mathbf{1} + \xi\mathbf{S}_1 + \xi^2\mathbf{S}_2 + \dots + \xi^{12}\mathbf{S}_{10},$$

with $l(\mathbf{S}) \leq 15$ and $r(\mathbf{1} - \mathbf{S}_i) = 3$, hence $\mathbf{S}_i = -\alpha - \alpha^2$ for at most two \mathbf{S}_i . If we first consider the case when $\mathbf{S}_i = \mathbf{1}$ for every i , then we just have the sum

$$\mathbf{1} + \xi + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \xi^6 + \xi^7 + \xi^8 + \xi^9 + \xi^{10} + \xi^{11} + \xi^{12}.$$

If $\mathbf{S}_i = -\alpha - \alpha^2$ for one such i then \mathbf{S} is similar to

$$-(\alpha + \alpha^2) + \xi + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \xi^6 + \xi^7 + \xi^8 + \xi^9 + \xi^{10} + \xi^{11} + \xi^{12}.$$

Finally if $\mathbf{S}_i = -\alpha - \alpha^2$ for two i then it is similar to

$$\mathbf{1} + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \xi^6 + \xi^7 + \xi^8 + \xi^9 + \xi^{10} + \xi^{11} - (\alpha + \alpha^2)(\xi + \xi^{12}).$$

Now if \mathbf{S} is not a minimal sum, then any minimal subsum of \mathbf{S} must satisfy the

theorem. At least one such subsum has at most seven terms, and therefore \mathbf{S} must involve one of the following sets of roots:

$$\begin{aligned} &\{\theta, \alpha\theta, \alpha^2\theta\}, \quad \{\theta, \beta\theta, \beta^2\theta, \beta^3\theta, \beta^4\theta\}, \quad \{\alpha\theta, \alpha^2\theta, \beta\theta, \beta^2\theta, \beta^3\theta, \beta^4\theta\}, \\ &\{\theta, \gamma\theta, \gamma^2\theta, \gamma^3\theta, \gamma^4\theta, \gamma^5\theta, \gamma^6\theta\}, \quad \{\theta, \theta\beta^2, \theta\beta^3, \theta\alpha\beta, \theta\alpha\beta^4, \theta\alpha^2\beta, \theta\alpha^2\beta^4\} \end{aligned}$$

for some root θ .

□

2.4.3 Rational Sums of Cosines of Rational Angles

Theorem 2.4.3.1. *Suppose we have at most seven distinct rational multiples of π , for which some rational linear combination of their cosines is rational but no proper subset has this property. If $\varphi \in (0, \pi)$ and all other angles are normalised to lie in $(0, \frac{\pi}{2})$, then the appropriate linear combination is proportional to one of the following:*

$$0 = \sum_{j=0}^2 \cos\left(\varphi + \frac{2j\pi}{3}\right), \quad \varphi \neq \frac{m\pi}{6} \quad (\text{i})$$

$$0 = \sum_{j=0}^4 \cos\left(\varphi + \frac{2j\pi}{5}\right), \quad \varphi \neq \frac{m\pi}{10}, \frac{n\pi}{15} \quad (\text{ii})$$

$$0 = \sum_{j=1}^2 \cos\left(\varphi + \frac{2j\pi}{3}\right) - \sum_{k=1}^4 \cos\left(\varphi + \frac{2k\pi}{5}\right), \quad \varphi \neq \frac{m\pi}{6}, \frac{n\pi}{10}, \frac{l\pi}{15} \quad (\text{iii})$$

$$0 = \sum_{j=0}^6 \cos\left(\varphi + \frac{2j\pi}{7}\right), \quad \varphi \neq \frac{m\pi}{14}, \frac{n\pi}{21} \quad (\text{iv})$$

$$0 = \cos(\varphi) + \cos\left(\varphi \pm \frac{2\pi}{5}\right) - \cos\left(\varphi \pm \frac{2\pi}{15}\right) + \cos\left(\varphi \pm \frac{7\pi}{15}\right) \quad (\text{v})$$

$$0 = \cos(\varphi) - \cos\left(\varphi \pm \frac{\pi}{5}\right) + \cos\left(\varphi \pm \frac{\pi}{15}\right) - \cos\left(\varphi \pm \frac{4\pi}{15}\right) \quad (\text{vi})$$

$$\frac{1}{2} = \cos \frac{\pi}{3} \quad (\text{vii})$$

$$\frac{1}{2} = \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} \quad (\text{viii})$$

$$\frac{1}{2} = \cos \frac{\pi}{5} - \cos \frac{\pi}{15} + \cos \frac{4\pi}{15} \quad (\text{ix})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{5} + \cos \frac{2\pi}{15} - \cos \frac{7\pi}{15} \quad (\text{x})$$

$$\frac{1}{2} = -\cos \frac{\pi}{15} + \cos \frac{2\pi}{15} + \cos \frac{4\pi}{15} - \cos \frac{7\pi}{15} \quad (\text{xi})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} \quad (\text{xii})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{2\pi}{21} - \cos \frac{5\pi}{21} \quad (\text{xiii})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} + \cos \frac{3\pi}{7} - \cos \frac{\pi}{21} + \cos \frac{8\pi}{21} \quad (\text{xiv})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{4\pi}{21} + \cos \frac{10\pi}{21} \quad (\text{xv})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} - \cos \frac{\pi}{21} + \cos \frac{2\pi}{21} - \cos \frac{5\pi}{21} + \cos \frac{8\pi}{21} \quad (\text{xvi})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{7} + \cos \frac{2\pi}{21} + \cos \frac{4\pi}{21} - \cos \frac{5\pi}{21} + \cos \frac{10\pi}{21} \quad (\text{xvii})$$

$$\frac{1}{2} = \cos \frac{3\pi}{7} - \cos \frac{\pi}{21} + \cos \frac{4\pi}{21} + \cos \frac{8\pi}{21} + \cos \frac{10\pi}{21} \quad (\text{xviii})$$

$$\frac{1}{2} = -\cos \frac{\pi}{21} + \cos \frac{2\pi}{21} + \cos \frac{4\pi}{21} - \cos \frac{5\pi}{21} + \cos \frac{8\pi}{21} + \cos \frac{10\pi}{21} \quad (\text{xix})$$

$$\frac{1}{2} = \cos \frac{\pi}{11} - \cos \frac{2\pi}{11} + \cos \frac{3\pi}{11} - \cos \frac{4\pi}{11} + \cos \frac{5\pi}{11} \quad (\text{xx})$$

$$\frac{1}{2} = \cos \frac{\pi}{11} + \cos \frac{3\pi}{11} - \cos \frac{4\pi}{11} + \cos \frac{5\pi}{11} - \cos \frac{5\pi}{33} + \cos \frac{16\pi}{33} \quad (\text{xxi})$$

$$\frac{1}{2} = \cos \frac{\pi}{11} - \cos \frac{2\pi}{11} + \cos \frac{3\pi}{11} + \cos \frac{5\pi}{11} - \cos \frac{\pi}{33} + \cos \frac{10\pi}{33} \quad (\text{xxii})$$

$$\frac{1}{2} = \cos \frac{\pi}{11} - \cos \frac{2\pi}{11} + \cos \frac{3\pi}{11} - \cos \frac{4\pi}{11} + \cos \frac{4\pi}{33} - \cos \frac{7\pi}{33} \quad (\text{xxiii})$$

$$\frac{1}{2} = \cos \frac{\pi}{11} - \cos \frac{3\pi}{11} - \cos \frac{4\pi}{11} + \cos \frac{5\pi}{11} + \cos \frac{2\pi}{33} - \cos \frac{13\pi}{33} \quad (\text{xxiv})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{11} + \cos \frac{3\pi}{11} - \cos \frac{4\pi}{11} + \cos \frac{5\pi}{11} + \cos \frac{8\pi}{33} + \cos \frac{14\pi}{33} \quad (\text{xxv})$$

$$\frac{1}{2} = \cos \frac{\pi}{13} - \cos \frac{2\pi}{13} + \cos \frac{3\pi}{13} - \cos \frac{4\pi}{13} + \cos \frac{5\pi}{13} - \cos \frac{6\pi}{13} \quad (\text{xxvi})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} + \cos \frac{3\pi}{7} - \cos \frac{3\pi}{35} + \cos \frac{4\pi}{35} - \cos \frac{11\pi}{35} - \cos \frac{17\pi}{35} \quad (\text{xxvii})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} - \cos \frac{2\pi}{7} - \cos \frac{\pi}{35} + \cos \frac{6\pi}{35} + \cos \frac{8\pi}{35} - \cos \frac{13\pi}{35} \quad (\text{xxviii})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{2\pi}{35} - \cos \frac{9\pi}{35} + \cos \frac{12\pi}{35} + \cos \frac{16\pi}{35} \quad (\text{xxix})$$

$$\frac{1}{2} = \cos \frac{\pi}{5} - \cos \frac{\pi}{35} + \cos \frac{4\pi}{35} + \cos \frac{6\pi}{35} - \cos \frac{9\pi}{35} - \cos \frac{11\pi}{35} + \cos \frac{16\pi}{35} \quad (\text{xxx})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{5} + \cos \frac{2\pi}{35} - \cos \frac{3\pi}{35} + \cos \frac{8\pi}{35} + \cos \frac{12\pi}{35} - \cos \frac{13\pi}{35} - \cos \frac{17\pi}{35} \quad (\text{xxxi})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} - \cos \frac{\pi}{21} + \cos \frac{8\pi}{21} - \cos \frac{\pi}{35} + \cos \frac{6\pi}{35} + \cos \frac{8\pi}{35} - \cos \frac{13\pi}{35} \quad (\text{xxxii})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{7} + \cos \frac{2\pi}{21} - \cos \frac{5\pi}{21} + \cos \frac{2\pi}{35} - \cos \frac{9\pi}{35} + \cos \frac{12\pi}{35} + \cos \frac{16\pi}{35} \quad (\text{xxxiii})$$

$$\frac{1}{2} = \cos \frac{3\pi}{7} + \cos \frac{4\pi}{21} + \cos \frac{10\pi}{21} - \cos \frac{3\pi}{35} + \cos \frac{4\pi}{35} - \cos \frac{11\pi}{35} - \cos \frac{17\pi}{35} \quad (\text{xxxiv})$$

$$\frac{1}{2} = \cos \frac{3\pi}{7} - \cos \frac{\pi}{21} + \cos \frac{8\pi}{21} + \cos \frac{2\pi}{35} - \cos \frac{9\pi}{35} + \cos \frac{12\pi}{35} + \cos \frac{16\pi}{35} \quad (\text{xxxv})$$

$$\frac{1}{2} = \cos \frac{\pi}{7} + \cos \frac{2\pi}{21} - \cos \frac{5\pi}{21} - \cos \frac{3\pi}{35} + \cos \frac{4\pi}{35} - \cos \frac{11\pi}{35} - \cos \frac{17\pi}{35} \quad (\text{xxxvi})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{7} + \cos \frac{4\pi}{21} + \cos \frac{10\pi}{21} - \cos \frac{\pi}{35} + \cos \frac{6\pi}{35} + \cos \frac{8\pi}{35} - \cos \frac{13\pi}{35} \quad (\text{xxxvii})$$

$$\frac{1}{2} = \cos \frac{\pi}{11} + \cos \frac{3\pi}{11} + \cos \frac{5\pi}{11} - \cos \frac{\pi}{33} - \cos \frac{5\pi}{33} + \cos \frac{10\pi}{33} + \cos \frac{16\pi}{33} \quad (\text{xxxviii})$$

$$\frac{1}{2} = \cos \frac{\pi}{11} - \cos \frac{2\pi}{11} + \cos \frac{5\pi}{11} - \cos \frac{\pi}{33} + \cos \frac{2\pi}{33} + \cos \frac{10\pi}{33} - \cos \frac{13\pi}{33} \quad (\text{xxxix})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{11} + \cos \frac{3\pi}{11} - \cos \frac{4\pi}{11} + \cos \frac{4\pi}{33} - \cos \frac{7\pi}{33} + \cos \frac{8\pi}{33} + \cos \frac{14\pi}{33} \quad (\text{xl})$$

$$\frac{1}{2} = \cos \frac{\pi}{11} - \cos \frac{2\pi}{11} - \cos \frac{4\pi}{11} + \cos \frac{2\pi}{33} + \cos \frac{4\pi}{33} - \cos \frac{7\pi}{33} - \cos \frac{13\pi}{33} \quad (\text{xli})$$

$$\frac{1}{2} = \cos \frac{3\pi}{11} - \cos \frac{4\pi}{11} + \cos \frac{5\pi}{11} - \cos \frac{5\pi}{33} + \cos \frac{8\pi}{33} + \cos \frac{14\pi}{33} + \cos \frac{16\pi}{33} \quad (\text{xlii})$$

$$\frac{1}{2} = \cos \frac{\pi}{11} + \cos \frac{3\pi}{11} - \cos \frac{4\pi}{11} + \cos \frac{4\pi}{33} - \cos \frac{5\pi}{33} - \cos \frac{7\pi}{33} + \cos \frac{16\pi}{33} \quad (\text{xliii})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{11} + \cos \frac{3\pi}{11} + \cos \frac{5\pi}{11} - \cos \frac{\pi}{33} + \cos \frac{8\pi}{33} + \cos \frac{10\pi}{33} + \cos \frac{14\pi}{33} \quad (\text{xliv})$$

$$\frac{1}{2} = \cos \frac{\pi}{11} - \cos \frac{2\pi}{11} + \cos \frac{3\pi}{11} - \cos \frac{\pi}{33} + \cos \frac{4\pi}{33} - \cos \frac{7\pi}{33} + \cos \frac{10\pi}{33} \quad (\text{xlv})$$

$$\frac{1}{2} = \cos \frac{\pi}{11} - \cos \frac{4\pi}{11} + \cos \frac{5\pi}{11} + \cos \frac{2\pi}{33} - \cos \frac{5\pi}{33} - \cos \frac{13\pi}{33} + \cos \frac{16\pi}{33} \quad (\text{xlvi})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{11} - \cos \frac{4\pi}{11} + \cos \frac{5\pi}{11} + \cos \frac{2\pi}{33} + \cos \frac{8\pi}{33} - \cos \frac{13\pi}{33} + \cos \frac{14\pi}{33} \quad (\text{xlvii})$$

$$\frac{1}{2} = \cos \frac{\pi}{13} + \cos \frac{3\pi}{13} - \cos \frac{4\pi}{13} + \cos \frac{5\pi}{13} - \cos \frac{6\pi}{13} - \cos \frac{7\pi}{39} - \cos \frac{19\pi}{39} \quad (\text{xlviii})$$

$$\frac{1}{2} = \cos \frac{\pi}{13} - \cos \frac{2\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} - \cos \frac{6\pi}{13} - \cos \frac{\pi}{39} + \cos \frac{14\pi}{39} \quad (\text{xlix})$$

$$\frac{1}{2} = \cos \frac{\pi}{13} - \cos \frac{2\pi}{13} + \cos \frac{3\pi}{13} - \cos \frac{4\pi}{13} + \cos \frac{5\pi}{13} - \cos \frac{5\pi}{39} + \cos \frac{8\pi}{39} \quad (\text{l})$$

$$\frac{1}{2} = \cos \frac{\pi}{13} - \cos \frac{2\pi}{13} + \cos \frac{3\pi}{13} - \cos \frac{4\pi}{13} - \cos \frac{6\pi}{13} + \cos \frac{2\pi}{39} - \cos \frac{11\pi}{39} \quad (\text{li})$$

$$\frac{1}{2} = \cos \frac{\pi}{13} - \cos \frac{2\pi}{13} - \cos \frac{4\pi}{13} + \cos \frac{5\pi}{13} - \cos \frac{6\pi}{13} + \cos \frac{4\pi}{39} - \cos \frac{17\pi}{39} \quad (\text{lii})$$

$$\frac{1}{2} = -\cos \frac{2\pi}{13} + \cos \frac{3\pi}{13} - \cos \frac{4\pi}{13} + \cos \frac{5\pi}{13} - \cos \frac{6\pi}{13} + \cos \frac{10\pi}{39} + \cos \frac{16\pi}{39} \quad (\text{liii})$$

Proof. For the proof we use Theorem 2.4.2.1. Using the equation $2 \cos \theta = e^{i\theta} + e^{-i\theta}$ we may interchange between rational sums of cosines and sums of roots of unity in which every root appears with the same coefficient as its complex conjugate. Therefore, from a vanishing sum that is stable under conjugation and has at most 15 terms, we obtain a rational linear combination of at most seven cosines in which we can normalise the angles to the range $(0, \frac{\pi}{2})$.

We may now apply Theorem 2.4.2.1. If $\{\theta, \alpha\theta, \alpha^2\theta\}$ or $\{\theta, \beta\theta, \beta^2\theta, \beta^3\theta, \beta^4\theta\}$ is involved then we have two cases. Firstly if at least one pair of the roots give the same cosine then we obtain equations, which normalise to

$$\cos \frac{\pi}{3} = \frac{1}{2} \quad \text{or} \quad \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} = \frac{1}{2}.$$

If none of the roots give the same cosine then we obtain (i) and (ii) respectively.

Similarly, if $\{\alpha\theta, \alpha^2\theta, \beta\theta, \beta^2\theta, \beta^3\theta, \beta^4\theta\}$ is involved and all roots correspond to distinct cosines then we have (iii). If at least two roots give the same cosine or one of the cosines is a multiple of $\frac{\pi}{3}$ then we obtain one of the equations (vii), ..., (xi).

If $\{\theta, \gamma\theta, \gamma^2\theta, \gamma^3\theta, \gamma^4\theta, \gamma^5\theta, \gamma^6\theta\}$ is involved and all roots correspond to distinct cosines then we have (vii). If at least two roots give the same cosine or one of the

cosines is a multiple of $\frac{\pi}{3}$ then we obtain one of the equations (xii),..., (xix).

If $\{\theta, \theta\beta^2, \theta\beta^3, \theta\alpha\beta, \theta\alpha\beta^4, \theta\alpha^2\beta, \theta\alpha^2\beta^4\}$ is involved and all roots correspond to distinct cosines then we have (v) or (vi). If at least two roots give the same cosine or one of the cosines is a multiple of $\frac{\pi}{3}$ then we obtain one of the equations (vii),..., (xi).

If the sum is similar to one of those displayed in part b) of Theorem 2.4.2.1, we may assume that it is exactly one of those listed, since each root appears as often as its complex conjugate. Taking all possibilities for the primitive roots involved, and normalising the resulting angles, we obtain the results (xiii),..., (liii). We need only consider those sums which involve **1**, as any sum which doesn't involve **1** will produce a linear combination of (xiii),..., (liii). For example, consider the sum

$$S = 1 + \gamma^2 + \gamma^3 + \gamma^4 + \gamma^5 - (\beta + \beta^2 + \beta^3 + \beta^4)(\gamma + \gamma^6).$$

We need to take all possibilities for the primitive roots involved. Changing the primitive fifth root β will not change the sum. However, we can take γ , γ^2 or γ^3 to be the primitive seventh root and obtain different sums. For the primitive root γ the sum is as above, but for γ^2 and γ^3 we have the sums

$$1 + \gamma + \gamma^3 + \gamma^4 + \gamma^6 - (\beta + \beta^2 + \beta^3 + \beta^4)(\gamma^2 + \gamma^5),$$

$$1 + \gamma + \gamma^2 + \gamma^5 + \gamma^6 - (\beta + \beta^2 + \beta^3 + \beta^4)(\gamma^3 + \gamma^4),$$

respectively. Taking $\beta = e^{\frac{2\pi i}{5}}$ and $\gamma = e^{\frac{2\pi i}{7}}$ we obtain the equations

$$\begin{aligned} 0 &= 1 + 2 \cos\left(\frac{4\pi}{7}\right) + 2 \cos\left(\frac{6\pi}{7}\right) - 4 \cos\left(\frac{2\pi}{5}\right) \cos\left(\frac{2\pi}{7}\right) - 4 \cos\left(\frac{4\pi}{5}\right) \cos\left(\frac{2\pi}{7}\right) \\ \Rightarrow \frac{1}{2} &= \cos\frac{\pi}{7} + \cos\frac{3\pi}{7} - \cos\frac{3\pi}{35} + \cos\frac{4\pi}{35} - \cos\frac{11\pi}{35} - \cos\frac{17\pi}{35}, \end{aligned}$$

$$\begin{aligned}
0 &= 1 + 2 \cos\left(\frac{2\pi}{7}\right) + 2 \cos\left(\frac{6\pi}{7}\right) - 4 \cos\left(\frac{2\pi}{5}\right) \cos\left(\frac{4\pi}{7}\right) - 4 \cos\left(\frac{4\pi}{5}\right) \cos\left(\frac{4\pi}{7}\right) \\
\Rightarrow \frac{1}{2} &= \cos\frac{\pi}{7} - \cos\frac{2\pi}{7} - \cos\frac{\pi}{35} + \cos\frac{6\pi}{35} + \cos\frac{8\pi}{35} - \cos\frac{13\pi}{35},
\end{aligned}$$

and

$$\begin{aligned}
0 &= 1 + 2 \cos\left(\frac{2\pi}{7}\right) + 2 \cos\left(\frac{4\pi}{7}\right) - 4 \cos\left(\frac{2\pi}{5}\right) \cos\left(\frac{6\pi}{7}\right) - 4 \cos\left(\frac{4\pi}{5}\right) \cos\left(\frac{6\pi}{7}\right) \\
\Rightarrow \frac{1}{2} &= -\cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{2\pi}{35} - \cos\frac{9\pi}{35} + \cos\frac{12\pi}{35} + \cos\frac{16\pi}{35},
\end{aligned}$$

respectively. □

Chapter 3

Ultra Parallel Triangle Groups

In this chapter we study the ultra parallel complex hyperbolic triangle group $[l_2, l_1, 0]$ and we use a ‘ping-pong’ method, similar to that used by Schwartz [29] and Wyss-Gallifent [34], to give a domain of discreteness. Wyss-Gallifent [34] studied the ultra parallel $[l, l, 2l]$ triangle groups and used this discreteness criterion to obtain partial results, which for $l = 0$ were similar to that of Goldman and Parker [10]. He also sampled some of the more general $[l_1, l_2, l_3]$ triangle groups and produced a similar result.

Wyss-Gallifent then studied the $[l, l, 0]$ triangle groups and by using this ‘ping-pong’ method approach gave a domain of discreteness. We will follow a similar route.

In Section 3.2 we introduce metric bisectors. A bisector is the equidistant surface between two distinct points in $\mathbf{H}_{\mathbb{C}}^2$. The intersection of a bisector with the boundary is a smooth surface in $\partial\mathbf{H}_{\mathbb{C}}^2$, we call such surfaces spinal spheres. We consider one particularly symmetric spinal sphere, called the *unit spinal sphere*. We then use the unit spinal sphere and look at reflections in the chains of the $[l_2, l_1, 0]$ triangle groups to use this ‘ping-pong’ method in the boundary $\partial\mathbf{H}_{\mathbb{C}}^2$ to find a domain of discreteness.

Remark 3.0.3.2. *The $[l_1, l_2, l_3]$ triangle groups Wyss-Gallifent [34] sampled are*

some of the triangle groups with

$$\cosh(l_3) = \cosh(l_1 + l_2) + \cosh(l_1 - l_2) - 1.$$

3.1 Discreteness Criterion

Let ι_1, ι_2 and ι_3 be reflections in the complex geodesics C_1, C_2 and C_3 respectively. Let Λ be the group generated by ι_1 and ι_2 and Γ is, as before, the group generated by ι_1, ι_2 and ι_3 .

Definition. Suppose Γ acts on a set S and $U_1, U_2, V \subset S$ with $U_1 \cap U_2 = \emptyset$ and $V \subsetneq U_1$. Then we say that $(U_1, U_2)_V$ is compressing for Γ if the following two conditions are met:

- $\iota_3(U_1) = U_2$.
- $g(U_2) \subsetneq V$ for all non-identity elements $g \in \Lambda$.

Remark 3.1.0.3. We also say Γ is compressing.

Definition. $(U_1, U_2)_V$ is said to be semi-compressing if only the second of these conditions is met.

Lemma 3.1.0.4. If Γ is compressing, then Γ is a discrete subgroup of $PU(2, 1)$.

Proof. Consider an element $g \in \Gamma$. There are four ‘types’ of g and each has an action on either U_1 or U_2 , which is isolated from the identity. Observe that g may be written as $g = \kappa_1 \iota_3 \kappa_2 \dots \iota_3 \kappa_n$, where all the κ_j are elements in Λ . Then notice the four ‘types’ of g and their effects in the following. For purposes of clarification assume that the leading and trailing κ_j ’s are non-identity.

(a) $\kappa_1 \iota_3 \kappa_2 \dots \iota_3 \kappa_n(U_2) \subsetneq V$

(b) $\kappa_1 \iota_3 \kappa_2 \dots \iota_3(U_1) \subsetneq V$

$$(c) \ \iota_3 \kappa_2 \dots \iota_3 \kappa_n(U_2) \subsetneq \iota_3(V)$$

$$(d) \ \iota_3 \kappa_2 \dots \iota_3(U_1) \subsetneq \iota_3(V)$$

In case (a), for any $x \in U_2$, $g(x) \in V$ and $(V \cap U_2) \subset (U_1 \cap U_2) = \emptyset$, so g is isolated from the identity. In case (b), take any $x \in U_1$ with $x \notin \overline{V}$, then $x \in U_1 \setminus \overline{V}$ and $g(x) \in V$, so g is isolated from the identity. We may follow a similar argument for cases (c) and (d) so that in all four cases, g is non-trivial and is isolated from the identity. Thus Γ is discrete. \square

3.2 Bisectors and Spinal Spheres

For a more detailed description of the following see [11, Chapter 5]. Unlike in the real hyperbolic space, there are no totally geodesic real hypersurfaces in $\mathbf{H}_{\mathbb{C}}^2$. An acceptable substitute are the metric bisectors.

Definition. Let $z_1, z_2 \in \mathbf{H}_{\mathbb{C}}^2$ be two distinct points. The bisector equidistant from z_1 and z_2 is defined as

$$\mathfrak{C}\{z_1, z_2\} = \{z \in \mathbf{H}_{\mathbb{C}}^2 \mid \rho(z_1, z) = \rho(z_2, z)\}.$$

Definition. The intersection of a bisector with the boundary of $\mathbf{H}_{\mathbb{C}}^2$ is a smooth hypersurface in $\partial\mathbf{H}_{\mathbb{C}}^2$ called a spinal sphere.

Definition. Let $\Sigma \subset \mathbf{H}_{\mathbb{C}}^2$ be the complex geodesic spanned by z_1, z_2 , then the spine of \mathfrak{C} equals

$$\sigma\{z_1, z_2\} = \mathfrak{C}\{z_1, z_2\} \cap \Sigma = \{z \in \Sigma \mid \rho(z_1, z) = \rho(z_2, z)\}.$$

That is, σ is the orthogonal bisector in Σ of the real geodesic segment joining z_1 and z_2 in Σ .

Theorem 3.2.0.5. *Let \mathfrak{C} , Σ and σ be as above and let $\Pi_\Sigma : \mathbf{H}_\mathbb{C}^2 \rightarrow \Sigma$ be the orthogonal projection onto Σ . Then*

$$\mathfrak{C} = \Pi_\Sigma^{-1}(\sigma) = \bigcup_{s \in \sigma} \Pi_\Sigma^{-1}(s)$$

Proof. [11, for proof, see Theorem 5.1.1] □

Definition. *The complex geodesics $\Pi_\Sigma^{-1}(s)$, for $s \in \sigma$, are called the slices of \mathfrak{C} .*

Definition. *We call the endpoints of the spine of \mathfrak{C} the vertices of the bisector \mathfrak{C} .*

Since the geodesic $\sigma \subset \mathbf{H}_\mathbb{C}^2$ is completely determined by the unordered pair $\partial\sigma \subset \partial\mathbf{H}_\mathbb{C}^2$ consisting of its endpoints, bisectors are completely parametrised by unordered pairs of distinct points in $\partial\mathbf{H}_\mathbb{C}^2$. We have associated to every bisector a geodesic (its spine). Conversely, if $\sigma \subset \mathbf{H}_\mathbb{C}^2$ is a geodesic, there exists a unique bisector $\mathfrak{C} = \mathfrak{C}_\sigma$ that has spine σ .

In $\mathbf{H}_\mathbb{C}^2$, no isometry exists whose fixed point set is a real hypersurface, since otherwise such a hypersurface would be totally geodesic. Instead for each slice S of a bisector \mathfrak{C} , inversion ι_S in S leaves \mathfrak{C} invariant, but only pointwise fixes the slice S . Inversion ι_S acts by reflection on the spine σ , interchanging the two endpoints and fixing the point $S \cap \sigma$.

Theorem 3.2.0.6. *Let $S \subset \mathbf{H}_\mathbb{C}^2$ be a complex geodesic and let ι_S denote inversion in S . Suppose $w_1, w_2 \in \partial\mathbf{H}_\mathbb{C}^2$. Then S is a slice of the bisector \mathfrak{C} having vertices w_1, w_2 if and only if ι_S interchanges w_1 and w_2 .*

Proof. For proof see [11, Theorem 5.2.1] □

An example of a spinal sphere which is a compact subset of \mathcal{H} is the unit spinal sphere, which is somewhat analogous to a unit sphere in the Heisenberg space. Consider the bisector \mathfrak{C} with vertices $(0, \pm 1) \in \mathcal{H}$, which corresponds to $(0, \pm i)$

in the ball model. In the ball model, the corresponding spine is the geodesic

$$\sigma = \{(0, it) \in \mathbb{B}^2 \mid -1 < t < 1\},$$

and the bisector is

$$\mathfrak{C} = \{(z, it) \in \mathbb{B}^2 \mid |z|^2 < 1 - t^2\}.$$

It is bounded by the spinal sphere $\partial\mathfrak{C}$ described by

$$|\zeta|^4 + v^2 = 1, \quad \text{where } (\zeta, v) \in \mathcal{H}.$$

The unit spinal sphere is shown below in Figure 3.1

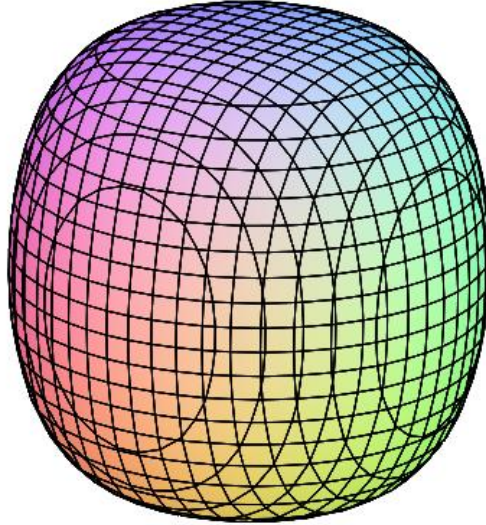


Figure 3.1: Unit spinal sphere in \mathcal{H}

3.3 $[l_2, l_1, 0]$ Triangle Groups

Let C_1 , C_2 and C_3 be the complex geodesics with respective polar vectors

$$c_1 = \begin{bmatrix} 1 \\ -\bar{z}_1 \\ \bar{z}_1 \end{bmatrix} \quad \text{and} \quad c_2 = \begin{bmatrix} 1 \\ \bar{z}_2 \\ -\bar{z}_2 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

where $z_1, z_2 \in \mathbb{C}$. Let $z_1 = r_1 e^{i\theta}$ and $z_2 = r_2 e^{-i\theta}$, where $r_1, r_2 \in \mathbb{R}^+$, $\theta \in [0, \pi/2)$.

The type of the triangle formed by C_1 , C_2 and C_3 is determined by

$$|\langle c_1, c_2 \rangle| = 1,$$

$$|\langle c_1, c_3 \rangle| = r_1,$$

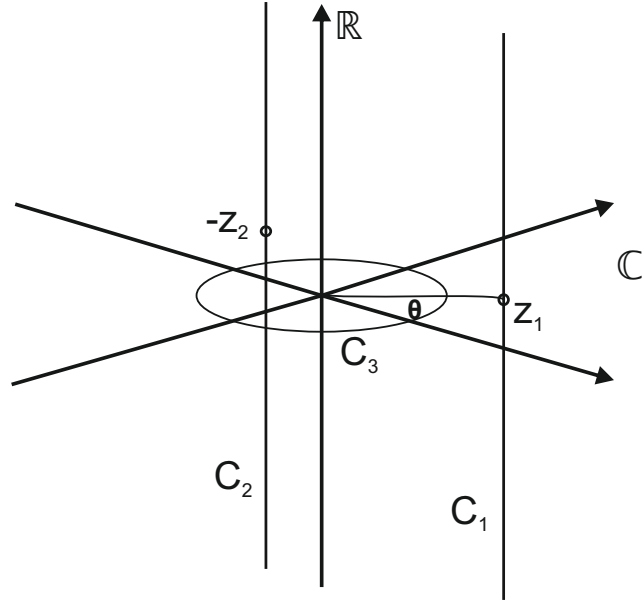
$$|\langle c_2, c_3 \rangle| = r_2.$$

We shall now consider the case when $r_1, r_2 > 1$, the triangle formed by C_1 , C_2 and C_3 is then an ultra-parallel $[l_2, l_1, 0]$ triangle, where $\cosh(l_k/2) = r_k$. The case when $l_1 = l_2$ was discussed in [34].

If we try and relate θ back to the angular invariant α , we find

$$\begin{aligned} \alpha &= \arg \left(\prod_{k=1}^3 \langle c_{k-1}, c_{k+1} \rangle \right) \\ &= \arg(-\bar{z}_1 z_2) \\ &= \arg(-r_1 r_2 e^{-2i\theta}) \\ &= \pi - 2\theta. \end{aligned}$$

We shall now revert from looking at reflections in the geodesics C_1 , C_2 and C_3 and instead talk about reflections in the chains C_1 , C_2 and C_3 , as described in Remark 1.1.3.2. If we look at the arrangement of the chains C_1 , C_2 and C_3 in \mathcal{H} ,

Figure 3.2: The chains in \mathcal{H}

the finite chain C_3 is the (Euclidean) unit circle in $\mathbb{C} \times \{0\}$, whereas C_1 and C_2 are vertical lines through z_1 and $-z_2$ respectively. Since $r_1, r_2 > 1$, the chains C_1 and C_2 lie outside the chain C_3 . This is depicted in Figure 3.2.

Suppose we have a vertical chain C_ζ which intersects $\mathbb{C} \times \{0\}$ at ζ with polar vector

$$c_\zeta = \begin{bmatrix} 1 \\ -\bar{\zeta} \\ \bar{\zeta} \end{bmatrix}.$$

Inversion in C_ζ is given by the matrix

$$\iota_{C_\zeta}(z) = \begin{pmatrix} 1 & -2\zeta & -2\zeta \\ -2\bar{\zeta} & 2|\zeta|^2 - 1 & 2|\zeta|^2 \\ 2\bar{\zeta} & -2|\zeta|^2 & -2|\zeta|^2 - 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

What effect does inversion in the vertical chain C_ζ have on another vertical chain, say C_ξ , where C_ξ is the vertical chain that passes through $(\xi, 0) \in \mathcal{H}$ and has polar vector

$$c_\xi = \begin{bmatrix} 1 \\ -\bar{\xi} \\ \bar{\xi} \end{bmatrix}?$$

We simply calculate

$$\begin{aligned} \iota_{C_\zeta}(z) &= \begin{pmatrix} 1 & -2\zeta & -2\zeta \\ -2\bar{\zeta} & 2|\zeta|^2 - 1 & 2|\zeta|^2 \\ 2\bar{\zeta} & -2|\zeta|^2 & -2|\zeta|^2 - 1 \end{pmatrix} \begin{bmatrix} 1 \\ -\bar{\xi} \\ \bar{\xi} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -\overline{(2\zeta - \xi)} \\ \overline{(2\zeta - \xi)} \end{bmatrix}, \end{aligned}$$

which is the polar vector of a vertical chain that intersects $\mathbb{C} \times \{0\}$ at $2\zeta - \xi$. Therefore if we have a vertical chain C_ξ , inversion in another vertical chain C_ζ rotates C_ξ as a set around C_ζ through π . There also may be a vertical translation on the chain itself.

Consider the unit spinal sphere U , discussed in section 3.2, with vertices $(0, \pm 1) \in \mathcal{H}$. Note that this is not a Euclidean sphere inside \mathcal{H} , but instead the set of points $U = \{(\zeta, v) \in \mathcal{H} : |\zeta|^4 + v^2 = 1\}$. Therefore the chain C_3 , the

horizontal unit circle, is a slice of the unit spinal sphere and thus inversion in C_3 leaves U invariant. An inversion in C_3 is given by the matrix

$$\iota_{C_3} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The point $(0,0) \in \mathcal{H}$ corresponds to $[0 : 1 : 1] \in \mathbf{H}_{\mathbb{C}}^2$ and the point $\infty \in \mathcal{H}$ corresponds to $[0, -1, 1] \in \mathbf{H}_{\mathbb{C}}^2$. Now

$$\begin{aligned} \iota_{C_3}([0 : 1 : 1]) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ &= [0 : 1 : -1] \\ &= [0 : -1 : 1]. \end{aligned}$$

Thus inversion in the chain C_3 interchanges the centre and ∞ in \mathcal{H} , so it must be that the inversion in C_3 switches the inside of U with the outside and leaves U invariant.

Theorem 3.3.0.7. *Let $0 \leq l_1, l_2 \in \mathbb{R}$. An ultra-parallel $[l_2, l_1, 0]$ triangle group is discrete for*

$$\sin \frac{\alpha}{2} \geq \frac{1}{r_1 + r_2},$$

where $r_k = \cosh(l_k/2)$.

Proof. We will first prove the following lemma.

Lemma 3.3.0.8. *Let C_1, C_2 and C_3 be the complex geodesics with respective polar*

vectors

$$c_1 = \begin{bmatrix} 1 \\ -\bar{z}_1 \\ \bar{z}_1 \end{bmatrix} \quad \text{and} \quad c_2 = \begin{bmatrix} 1 \\ \bar{z}_2 \\ -\bar{z}_2 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

where $z_1, z_2 \in \mathbb{C}$ and $|z_1|, |z_2| \geq 1$. Let ι_k denote inversion in C_k , let j_1, j_2 be the rotational components of ι_1, ι_2 in \mathbb{C} respectively. That is, j_k is the rotation around z_k through π . Let Γ be the group generated by all three inversions and Λ be the group generated by just ι_1, ι_2 . If $|g(0)| \geq 2$ for all non-trivial $g \in \langle j_1, j_2 \rangle$, then the group Γ is discrete.

Proof. We shall use the compressing criteria described above and the fact that a compressing triangle group is discrete.

Let $U = \{(z, y) \in \mathcal{H} : |z|^4 + y^2 = 1\}$ again be the unit spinal sphere in \mathcal{H} , with vertices at $(0, \pm 1)$. As shown above U contains the chain C_3 and inversion in C_3 preserves U and switches its inside and outside.

We choose U_1 to be the part of $\mathcal{H} \setminus U$ outside U containing ∞ , and U_2 to be the part inside containing the origin. Clearly we have $U_1 \cap U_2 = \emptyset$ and $\iota_3(U_1) = U_2$. Therefore if we find a subset $V \subsetneq U_1$ such that $g(U_2) \subsetneq V$ for all non-identity $g \in \Lambda$, then we have proven that Γ is compressing and hence discrete.

Inversion in a vertical chain acts on all the other vertical chains simply by a rotation of π radians. Some vertical translation of any individual inverted chain will also generally occur, but the chain is setwise sent to the vertical chain rotated by π radians around the chain we are inverting in.

Let W be the set of vertical chains through all $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. Let

$$\begin{aligned} W_1 &= \{(\zeta, v) : |\zeta| > 1\} \\ \text{and} \quad W_2 &= \{(\zeta, v) : |\zeta| < 1\}. \end{aligned}$$

We have that $U_2 \subset W_2$ and so $g(U_2) \subset g(W_2)$ for all non-identity $g \in \Lambda$.

W_2 is a union of vertical chains, elements of Γ map vertical chains to vertical chains, hence $g(W_2)$ is a union of vertical chains for any $g \in \Gamma$. If we look at the images $g(W_2)$ for $g \in \Lambda$, we can ignore the vertical translation that the inversion has on W_2 as we know $g(U)$ will lie somewhere inside $g(W_2)$. Therefore we can just simply look at the intersection of the images of W_2 with $\mathbb{C} \times \{0\}$ and if they do not intersect then for sure the images of U_2 will not intersect. Each image of W_2 intersects $\mathbb{C} \times \{0\}$ in a disk. Elements of Λ move the unit circle around in \mathbb{C} by rotations of π radians around z_1 and z_2 . Provided that the interior of the unit circle is mapped completely off itself by all non-identity $g \in \Lambda$, then the same is true for W_2 and hence for U_2 . We can then choose V to be the union of all the images of U_2 . We can be sure that $V \neq U_1$ since V is missing all the images of $W_2 \setminus U_2$.

We are therefore only left to find what is required to be sure that the interior of the unit circle is mapped off itself by non-identity elements of the group generated by j_1 and j_2 . Since the radius of the circle is preserved under rotations, it suffices to show that the origin is moved to a distance of at least 2 by such non-trivial elements in $\langle j_1, j_2 \rangle$. This is precisely the condition in Lemma 3.3.0.8. \square

In general, rotating a point $\beta \in \mathbb{C}$ in another point $\gamma \in \mathbb{C}$ returns the point $2\gamma - \beta$. The inversions are of order two so we may ignore when we have a composition of the same inversion. Consider applying ι_1 then ι_2 then ι_1 and so on, to the origin. Starting with ι_2 will lead to a similar argument. Then we have the following possibilities

$$\begin{aligned}
 (j_2 j_1)^n(0) &= -2n(z_1 + z_2) \\
 &= -2n(r_1 e^{i\theta} + r_2 e^{-i\theta}), \\
 j_1(j_2 j_1)^n(0) &= 2z_1 + 2n(z_1 + z_2) \\
 &= 2r_1 e^{i\theta} + 2n(r_1 e^{i\theta} + r_2 e^{-i\theta}),
 \end{aligned}$$

where $n \in \mathbb{Z}_+$.

Now, for the interior of the unit circle to be mapped off itself we need

$$|(j_2 j_1)^n(0)| \geq 2,$$

for all $n > 0$. Now if $|\operatorname{Re}\{(j_1 j_2)^n(0)\}| \geq 2$ then clearly $|(j_2 j_1)^n(0)| \geq 2$. Hence we assume that

$$\begin{aligned} |\operatorname{Re}\{(j_2 j_1)^n(0)\}| &\geq 2 \\ \Leftrightarrow |-2n(r_1 + r_2) \cos \theta| &\geq 2 \\ \Leftrightarrow |n(r_1 + r_2) \cos \theta| &\geq 1 \\ \Rightarrow \cos \theta &\geq \frac{1}{r_1 + r_2}, \quad \text{is true for all integer } n > 0. \end{aligned}$$

But if this criterion is met, then automatically $|j_1(j_2 j_1)^n(0)| \geq 2$ as

$$\begin{aligned} \operatorname{Re}\{(j_1(j_2 j_1)^n(0))\} &= 2r_1 \cos \theta + 2n(r_1 + r_2) \cos \theta \\ \text{and} \quad \theta &\in [0, \pi/2). \end{aligned}$$

It follows that Γ is discrete for $\cos \theta \geq \frac{1}{r_1 + r_2}$. Using the substitution,

$$\theta = \frac{\pi}{2} - \frac{\alpha}{2},$$

we obtain what is required, that an ultra-parallel $[l_2, l_1, 0]$ triangle group is discrete for

$$\sin \frac{\alpha}{2} \geq \frac{1}{r_1 + r_2},$$

where $r_k = \cosh(l_k/2)$.

□

Remark 3.3.0.9.

$$\sin \frac{\alpha}{2} \geq \frac{1}{r_1 + r_2},$$

is equivalent to

$$\tan \theta \leq \sqrt{(r_1 + r_2)^2 - 1}.$$

Remark 3.3.0.10. *Theorem 3.3.0.7 is sufficient but not necessary. For example when $l_1 = l_2 = 0$, we obtain that the ideal triangle group is discrete for $\tan(\theta) \leq \sqrt{3}$. But as conjectured by Goldman and Parker [10] and proved by Schwartz the ideal triangle group is still discrete for larger values of $\tan \theta$, namely if and only if $\tan \theta \leq \sqrt{125/3}$.*

Remark 3.3.0.11. *We followed a similar proof as used by Wyss-Gallifent [34] but in a more general case. He studied the case when $r_1 = r_2$ and proved that an ultra-parallel $[l_1, l_1, 0]$ triangle group is discrete for*

$$\sin \frac{\alpha}{2} \geq \frac{1}{2r_1}$$

Chapter 4

Brief Summary

In this thesis, the first main study involved looking at complex hyperbolic triangle groups, when both the elements $w_A = \iota_3 \iota_2 \iota_3 \iota_1$ and $w_B = \iota_1 \iota_2 \iota_3$ were elliptic and determining whether the group was discrete or not. For the triangle group to be discrete both w_A and w_B would have to be of finite order.

For a general complex hyperbolic triangle group with angles $(\varphi_1, \varphi_2, \varphi_3)$, the question of whether the two elliptic elements could be of finite order reduces to solving

$$\begin{aligned} 1 &= \cos(2\varphi_1 + 2\varphi_2) + \cos(2\varphi_1 - 2\varphi_2) - \cos(2\varphi_3) \\ &\quad - \cos x - \cos y - \cos z - \cos \frac{2\pi}{n}, \end{aligned} \tag{2.1.3}$$

where $n \in \mathbb{Z}$, $x, y, z \in \mathbb{Q}\pi$. Conway and Jones [2] classified all rational linear combinations of at most four cosines of rational multiples of π which is rational and then Parker [21] extended the classification to all rational linear combinations of at most six cosines of rational multiples of π which vanish. Therefore to use work that has already been done we look at certain types of triangles with a relation between the three angles so that the equation reduces in length.

The first example we studied were the complex hyperbolic triangle groups with

signature $(p, p, 2p)$. For triangle groups of this type the equation reduces to

$$0 = \cos \frac{\pi}{p} - \cos \frac{4\pi}{p} + \cos x + \cos y + \cos z + \cos \frac{2\pi}{n}.$$

Thus we could use the list given by Parker to prove:

Theorem (2.2.0.19). *The complex hyperbolic triangle group $\Gamma(p, p, 2p; n)$ is non-discrete for $p \in \mathbb{Z}$, $p \geq 13$ when w_B is regular elliptic.*

The next class of triangle groups that we studied were those with angles $(\varphi_1, \varphi_2, \varphi_3)$ satisfying $\varphi_1 = \varphi_2 + \varphi_3$. This reduces the question of determining when w_A and w_B are both finite elliptic to solving

$$1 = \cos(2\varphi_1 + 2\varphi_2) - \cos x - \cos y - \cos z - \cos \frac{2\pi}{n}.$$

Using the results of Conway and Jones and studying $(p, 2p, 2p)$ and $(p, 3p/2, 3p)$ triangle groups in detail we prove the following:

Theorem (2.3.0.38). *The complex hyperbolic triangle group $\Gamma(p_1, p_2, p_3; n)$ with $\frac{\pi}{p_1} = \frac{\pi}{p_2} + \frac{\pi}{p_3}$ is non-discrete for all $p_1 \in \mathbb{Z}$, $p_1 \geq 12$ when w_B is regular elliptic.*

Therefore in our search, we found no discrete triangle groups of type B . There are known triangle groups that contradict the Schwartz conjecture, examples would be the $(18, 18, 18; 18)$ triangle group and a deformed $(14, 14, 14)$ triangle group found by Parker.

To determine whether a general complex hyperbolic triangle group, with two elements w_A and w_B both elliptic, is discrete or not we would need to extend the results of Conway and Jones and Parker to solve an equation of a rational linear combination of seven cosines of rational multiples of π with a rational sum. This is precisely what we did in section 2.4. Using similar steps as both Conway and

Jones and Parker we produce a finite list of sums that satisfy

$$A_0 = \sum_{i=1}^7 A_i \cos(\pi a_i), \quad \text{where } A_i, a_i \in \mathbb{Q}.$$

Hence, with our list, it is possible to check for any solutions to equation (2.1.3) for a general complex hyperbolic triangle. Therefore in principle, one can sieve through all non-discrete triangle groups of this type and leave only those, if any, that are a suitable candidate to be discrete. The best approach to this would be to use a computer program but we leave this as an open problem.

Next our attentions turned to the ultra-parallel triangle groups and in particular those groups of signature $[l_2, l_1, 0]$. We introduced a compression property and used it to give a domain of discreteness:

Theorem (3.3.0.7). *Let $0 \leq l_1, l_2 \in \mathbb{R}$. An ultra-parallel $[l_2, l_1, 0]$ triangle group is discrete for*

$$\sin \frac{\alpha}{2} \geq \frac{1}{r_1 + r_2},$$

where $r_k = \cosh(l_k/2)$.

Comparing with the results in the ideal triangle group case by Goldman and Parker and by Schwartz, we see that the discreteness condition in Theorem 3.3.0.7 is sufficient, but not necessary, i.e. we expect the parameter interval that corresponds to discrete groups to be larger than the parameter interval in Theorem 3.3.0.7. Recall that the condition in Theorem 3.3.0.7 was chosen to ensure that the projections of the images of the unit spinal sphere were disjoint. One way to improve the result of Theorem 3.3.0.7 would be to work with the images of the unit spinal sphere themselves rather than with their projections.

Bibliography

- [1] A. Borel, Harish-Chandra, Arithmetic Subgroups of Algebraic Groups, *Annals of Mathematics*. **75**, (1962), p. 485-535
- [2] J. H. Conway, A. J. Jones, Trigonometric Diophantine Equations (On Vanishing Sums of Roots of Unity), *Acta Arithmetica*. **30**, (1976), p. 229-240.
- [3] K. Corlette, Archimedean Superrigidity and Hyperbolic Geometry, *Annals of Mathematics*. **135**, (1992), p. 165-182
- [4] P. Deligne, G. D. Mostow, Monodromy of Hypergeometric Functions and Non-lattice Integral Monodromy, *Publ. Math. I.H.E.S.* **63**, (1986), p. 5-89
- [5] M. Deraux, Dirichlet Domains for the Mostow Lattices, *Experimental Mathematics*. **14**, (2005), p. 467-490
- [6] M. Deraux, Deforming the \mathbb{R} -Fuchsian $(4, 4, 4)$ Triangle Group into a Lattice, *Topology*. **45.6**, (2006), p. 989-1020
- [7] M. Deraux, E. Falbel, J. Paupert, New Constructions of Fundamental Polyhedra in Complex Hyperbolic Space, *Acta Arithmetica*. **194**, (2005), p. 155-201
- [8] M. Deraux, J. Paupert, J. R. Parker, Census of the complex hyperbolic sporadic groups, *Experimental Mathematics*. **20**, (2011), p. 467-486
- [9] E. Falbel, J. R. Parker, The Geometry of the Eisenstein-Picard Modular group, *Duke Math. J.* **131.2**, (2006), p. 249-289

- [10] W. M. Goldman, J. R. Parker, Complex Hyperbolic Ideal Triangle Groups, *J. reine angew. Math* **425**, (1992), p. 71-86.
- [11] W. M. Goldman, *Complex Hyperbolic Geometry*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, (1999).
- [12] M. Gromov, I. Piatetski-Shapiro, Nonarithmetic Groups in Lobachevsky Spaces, *Publ. Math. I.H.E.S* **66**, (1988), p.93-103
- [13] M. Gromov, R. Schoen, Harmonic Maps into Singular Spaces and p-adic superrigidity for Lattices in Groups of Rank One, *Publ. Math. I.H.E.S* **76**, (1992), p. 165-246
- [14] C. H. Grossi, On the Type of Triangle Groups, *Geom. Dedicata*. **130**, (2007), p. 137-148
- [15] S. Kamiya, J. R. Parker, J. M. Thompson, Notes on Complex Hyperbolic Triangle Groups, *Conformal Geometry and Dynamics*. **14**, (2010), p. 202-218.
- [16] S. Kamiya, J. R. Parker, J. M. Thompson, Non-discrete Complex Hyperbolic Triangle Groups of Type $(n, n, \infty; k)$, *Canadian Mathematical Bulletin*. **55**, (2012), p. 329-338.
- [17] F. Klein, Neue Beiträge zur Riemannschen Functionentheorie, *Math. Ann.* **21**, (1883), p. 141-218
- [18] G. Margulis, Arithmeticity of the Irreducible Lattices in the Semisimple Groups of Rank Greater than One, *Invent. Math.* **76**, (1984), p. 93-120
- [19] B. Maskit, Construction of Kleinian groups, *Proceedings of Conference on Complex Analysis, Minneapolis*. (1964), p. 281-296.

- [20] G. D. Mostow, On a Remarkable Class of Polyhedra in Complex Hyperbolic Space, *Pacific Journal of Mathematics*. **86**, (1980), p. 171-276
- [21] J. R. Parker, *2-Generator Möbius Groups*, Ph.D. Thesis, University of Cambridge, (1989), p. 207-216.
- [22] J. R. Parker, Cone Metrics on the Sphere and Livné Lattices, *Acta Mathematica*. **196.1**, (2006), p. 1-64.
- [23] J. R. Parker, Unfaithful Complex Hyperbolic Triangle Groups, I: Involutions, *Pacific Journal of Mathematics*. **238**, (2008), p. 145-169.
- [24] J. R. Parker, Complex Hyperbolic Lattices, *Discrete Groups and Geometric Structures*, ed K.Dekimpe et al. *Contemporary Mathematics*. **501**, (2009), p. 1-42.
- [25] J. R. Parker, J. Paupert, Unfaithful Complex Hyperbolic Triangle Groups, II: Higher Order Reflections, *Pacific Journal of Mathematics*. **239**, (2009), p. 357-389.
- [26] B. Poonen, M. Rubinstein, The Number of Intersection Points Made by the Diagonals of a Regular Polygon, *SIAM Journal on Discrete Mathematics*. **11.1**, (1998), p. 135-156
- [27] A. Pratoussevitch, Traces in Complex Hyperbolic Triangle Groups, *Geom. Dedicata*. **111**, (2005), p. 159-185.
- [28] A. Pratoussevitch, Non-discrete Complex Hyperbolic Triangle Groups of type (m, m, ∞) , *Bulletin of the LMS* **43**, (2011), 359-363.
- [29] R. E. Schwartz, Ideal Triangle Groups, Dented Tori, and Numerical Analysis, *Annals of Mathematics*. **153**, (2001), pp. 533-598

- [30] R. E. Schwartz, Complex Hyperbolic Triangle Groups, *Proceedings of the International Congress of Mathematicians, Vol. II*, Beijing, Higher Ed. Press, (2002), p. 339-349.
- [31] R. E. Schwartz, Real Hyperbolic on the Outside, Complex Hyperbolic on the Inside, *Inventiones Mathematicae*. **151**.2, (2003), p. 221-295
- [32] K. Takeuchi, Arithmetic Triangle Groups, *Journal of the Mathematical Society of Japan*. **29**, (1977), p. 91-106.
- [33] J. M. Thompson, *Complex Hyperbolic Triangle Groups*, Ph. D. Thesis, University of Durham, (2010).
- [34] J. O. Wyss-Gallifent, *Complex Hyperbolic Triangle Groups*, Ph. D. Thesis, University of Maryland, (2000).